HINTS

ON

Teaching Arithmetic

H. S. MacLEAN

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TORONTO

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TEACHING ARITHMETIC

BY

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PREFACE.

As indicated by the title of this manual, no attempt has been made to enter into an exhaustive discussion of the psychological principles underlying the teaching of Arithmetic. The aim has been, rather, to present in brief form, and in as practical a manner as possible, what have appeared to the author to be the most important features of the subject, viewed from the teacher's standpoint.

The early pages are devoted chiefly to the consideration of such fundamental questions as seem to have the most direct bearing on practical school work. An effort is made, whether successful or not, to outline what may be done in the school-room during consecutive periods. Lastly, suggestions are offered on points where mistakes are likely to occur in teaching.

This book, containing the substance of talks given at institutes, and in the Manitoba Normal School, is humbly submitted for the consideration of teachers generally, with the hope that it may prove to be of some little value to, at least, the younger members of the profession.

CONTENTS.

HINTS ON TEACHING ARITHMETIC.

1. Importance as a Subject of Study.

Arithmetic is one of the few subjects whose right to a place on the Public School programme of studies has never been seriously questioned. Its adaptability to the different stages of mental development, its suitability as a means of training in concentration, accuracy, rapidity and logical arrangement of thought, its tendency to secure clearness, exactness and conciseness of expression, and its usefulness in the every day business of life are all so evident as to give it an undisputed right to a high position among school subjects.

The material world presents objects of endless variety which invite the attention of man. The human mind reacting on these ascribes to them qualities of form, size, etc., and classifies them according to resemblances and differences of such qualities. But these objects manifest diversity of form and size, and they are acted upon by forces. Hence the necessity for accurate quantitative measurement arises. A standard of measurement is determined and the relation between it and the whole quantity measured is established. This quantity relation, which is the product of the action of the mind, is termed number. The study of number is, therefore, necessary in order that man may be brought into harmony with his environment. Besides this, a knowledge of number is essential in all the details of actual life, for there could be no progress in science, manufactures, or commerce, without exact quantitative measurement.

The conditions of existence since the very earliest advances were made in civilization have rendered it imperative that material objects, as well as the forces acting upon them, should be subject to limitation. The necessity for estimating, weighing and measuring such objects and forces may be regarded as the prime motive impelling the individual to exercise mental power in quantitative determination. An indefinite whole quantity is presented; the mind feels the necessity for defining it. This fact has an important bearing on teaching number, for it points directly to a true basis of interest. The child passes through stages of growth corresponding, in the main, to those through which the race has passed.

2. Relation to Other Subjects.

Arithmetic bears a highly important relation to other subjects of study. By means of number we can give definite size to a continent, we can compare the heights of its mountain systems, we can measure its rivers and river basins, and we can estimate with a degree of accuracy the educational, social and commercial progress of its inhabitants. But it is unnecessary to multiply illustrations; astronomy, meteorology, chemistry, botany, physics, etc., will at once suggest many. Indeed, the natural sciences would forever have remained in a state of infancy but for the aid of mathematical calculation. Why is this the case? Simply for the reason that without number well-defined notions of size, distance, motion, etc., could never be attained. As arithmetic is so closely related to other subjects, would it not be well in the school-room not to divorce it entirely from them but rather to make it contribute its due share towards their advancement? A large proportion of the exercises in arithmetic should be closely connected with the daily experiences of the pupil, whether in the school-room, in the city, or on the farm.

In endeavouring to correlate arithmetic with other subjects,

we must guard against the opposite error of making it wholly subservient to them. The arithmetic lesson should be a lesson on arithmetic, and not on geography, botany or physiology. Arithmetic must be taught in accordance with the "inherent immortal rationality" of the subject. An attempt to teach several subjects in one lesson generally results in teaching nothing. The well-being of the pupil demands "concentration" of effort on the subject in hand, whatever that may be. There need be no conflict of opinion, however, between the teacher who advocates the correlation of studies and the teacher who insists on teaching one thing at a time. Both are right; they are simply viewing the matter from different standpoints.

3. Number a Purely Mental Conception.

Number is not an inherent quality of the material objects or groups of objects, forces, etc., that are measured numerically, but it is the product of the mind's action in determining their exact limitations as to quantity. This becomes obvious when we consider that the measure of any definite quantity will depend altogether on the unit of measurement which the mind chooses.

Objects and groups of objects are perceived by the senses, but not quantity relations. The presence of objects is a necessary condition—but only a condition—for occasioning the mental activity that results in establishing a quantity relation between a unit of measurement and a whole or aggregate. The sense concepts corresponding to objects presented, furnish the material upon which the mind operates in determining numerical relations.

Number is always abstract. There can be no such thing as concrete number. The unit may be particular or limited in its application, as in 10 pears, 10 yards, etc., or it may be general

or unlimited, as in 10. But no matter what the character of the unit may be, an abstraction must be made, otherwise, there can be no proper conception of number. In teaching, this important fact must not be forgotten.

Development of the Idea of Number.

It has been already stated that number is a purely mental conception, and that objects in space merely furnish the occasion of the mental action which results in establishing definite quantity relations. To make this clearer, and also to pave the way for the consideration of what follows, it may be well to endeavour to trace approximately the development of the idea of number. The steps may be roughly outlined as follows :

(i) Observing objects in space.

(ii) Separating in thought the idea of quantity from other observed qualities, as, form, colour, etc., and fixing the attention on quantity to the exclusion of other characteristics.

(iii) Perceiving differences of quantity. It is on this that numerical measurement is conditioned. At an early age such adjectives as much, little, large, small, long, short, many, few, etc., are used intelligently by children showing that comparisons of quantities are made by them.

(iv) Observing that a quantity may be separated into like parts (units), and that those parts when combined make up the whole.

(v) Giving to each act of attention in combining or separating the parts its place in the whole series of acts performed. This is what gives rise to the idea of ordinal number, as first, second, third, etc.

(vi) Synthesizing the different acts so as to form a complete whole. By this means the idea of cardinal number, as one, two, three, etc., is gained.

If we accept the foregoing as correct, we are in a position to mark out the lines along which progress in number must be made. Beginning with an undetermined quantity which is seen to be capable of being measured, the child separates it—roughly at first but with sufficient accuracy to satisfy him at this early stage—into parts, and by counting these he defines the whole. To his original percept of the whole as a quantity extended in *space* he has added the new element of *time.* The undetermined whole has become to him a numbered quantity. But as he gains strength intellectually greater exactness is demanded. The indefinite units by which he first measured the whole must become more and more definite, otherwise numbering can no longer afford a mental stimulus for him. Well defined, *measured* units now become a necessity, for without them the higher thought processes of number would be impossible. It is thus seen that the direction of progress in number is determined by the conditions of mental growth.

There are many other intermediate steps which are more or less difficult to determine. According to Perez,* M. Houzeau says: "Children at first can only distinguish between a single object and plurality. At eighteen months they can tell the difference between one, two, and several. At about three, or a little sooner, they have learnt to understand one, two, and four—two times two. It is scarcely ever till a later age that they can count the regular series, one, two, three, four; and here they stop for a long time."

4. What the Idea of Number Includes.

In every complete idea of number there are three elements.

(i) A unit of measurement.

(ii) An aggregate or whole quantity measured.

(iii) A quantity relation (number).

Any two of these determine the other. If either the unit

*Perez, *The First Three Years of Childhood*—p. 185.

or the quantity relation is unknown or not fully comprehended, the aggregate cannot be definitely fixed. For example, 20 square feet will not convey its full meaning to a pupil who has not a clear conception of the unit square foot, even if he knows the measure (20) of the aggregate 20 square feet. The same may be said in the case where the unit is well understood, but the measure of the aggregate (number) is not. From these considerations we can easily see that in teaching arithmetic, units themselves, as well as the numbering and relating of units, demand the careful attention of the teacher.

5. Units of Measurement.

By the term unit, in numbering, we mean one of the like things of a unity on which the attention is fixed as the basis of measurement. For example, when we speak of 12 pears the unit which we have in mind is one pear; in 12 square feet it is one square foot, and in $\$\frac{12}{25}$ it is one twenty-fifth of one dollar.

Any aggregation of units considered as one whole is termed a unity. It is evident that if any unity be regarded as a standard of measurement for like things, it then becomes the unit. Three feet may be taken as the unit of 12 feet, in which case the measure is 4 instead of 12. Again, if we take $\frac{1}{4}$ of a foot as the unit, the measure of 12 feet becomes 48. The measure, therefore, of any quantity depends altogether on the unit which the mind fixes upon as the basis of measurement. Before an aggregate can be measured the unit must be clearly seen. Much of the work in arithmetic consists in transforming quantities so that they can be measured by particular units and *vice versa*.

Sometimes it is convenient to take one unit and sometimes another. This will depend on the whole quantity measured or on the purpose for which the measurement is made. For

example, 1,000 is more easily thought of as 10 hundreds than as 1,000 ones. We can count 120 people in a room more readily by threes than by ones, and we can measure off 660 feet much more quickly by using a $16\frac{1}{2}$ foot pole, or 66 foot chain as the unit than we can by means of a foot rule. Again, it may be necessary to consider a part of a whole thing which we have in mind as the unit. We may wish to purchase $\frac{3}{4}$ of an acre of ground. Here the unit is $\frac{1}{4}$ of an acre. It is thus seen that the unit may be increased or diminished to any extent, at will, in order to meet the requirements of any particular case. The pupil should be given practice in selecting different units by which aggregates may be measured. He should also be led to see the necessity for changing from one unit to another.

Units may be divided into two kinds, concrete and abstract. A concrete unit is one which may be used to measure only a particular kind of quantity, as in 12 horses, 25 yards, etc. An abstract unit is one that is thought of as applicable to the measurement of any quantity whatever, as in 12.

Concrete units may be undefined as to quantity, as in 6 books, 16 houses, etc., or they may be well defined standards of measurement, as in 12 yards, 100 ohms, etc.

A fractional unit is one which is dependent for its value on its relation to some other unit. When we speak of $\frac{2}{3}$ of one yard we have in mind the unit $\frac{1}{3}$ of one yard, which is clearly dependent for its meaning on the unit one yard, to which it is related. In fact the fractional number $\frac{1}{3}$ does nothing more or less than express the ratio between the fractional unit and the prime unit from which it is derived.

What is the purpose of considering units as a factor in determining a course in number?

To discover their degree of definiteness in themselves in

order that we may proceed from the vague to the definite in accordance with the natural progress of thought. The order then is, (*a*) Concrete units of single objects which do not represent any fixed quantity, as block, pin, etc. An aggregate determined by such units gives rise to counting, and nothing more definite. (*b*) Units composed of numbered groups of such objects. Here the unit, though still somewhat indefinite, is limited to a degree as it is a numbered unity. (*c*) Fixed standards of measurement. These are well defined, and their use in measuring aggregates leads to the highest conception of measurement, viz., ratio, by directing the mind not only to the numerical value of the aggregate, but also to its quantitative value, as compared with that of the fixed unit. (*d*) Fractional units. Such units are completely defined in so far as their relation to the unit from which they are derived is concerned. The absolute definiteness of a fractional unit will depend entirely on the degree of definiteness of the related unit, as $\frac{1}{3}$ of a line, $\frac{1}{3}$ of 6 feet, etc.

As the simplest fraction expresses a relation based on a comparison of quantities, it is clear that fractions should not be introduced until the pupil has made considerable progress in relating quantities. The fact that many good teachers advocate the introduction of fractions at the very commencement of a course in number is manifestly the result of a difference of opinion as to the meaning of the term fraction.

In teaching a pupil to number, none but familiar objects, unattractive in themselves, should be used. To present objects as toys, flowers, etc., which possess striking characteristics may prove interesting to the pupil but the interest will be centered on the qualities of the objects, and consequently it will be foreign to the purpose of a lesson on number. As has already been shown, the pupil must lose sight of the distinctive qualities of the objects making up a group before

they can be regarded as units by which the group may be numbered. The pupil should be conscious of space limitation sufficient to distinguish one object from another and nothing more, in order that the best condition for numbering may be fulfilled. The less definite the unit is, so long as it is seen to be a unit, the more will the attention be fixed on *numbering* as a means of defining the aggregate. That is the reason why it is easier for the pupil to count objects by ones than by twos, threes, etc. As soon as the unit becomes definitely measured the difficulty of numbering is increased, because greater exactness of thought is demanded. This being the case, it is easily seen that units which themselves represent fixed measurements should not be presented until the pupil has gained some power in numbering. After he is once able to deal with such units, the measurement of aggregates by them will greatly increase his power, and his knowledge of number.

In introducing such measured units as inch, foot, yard, pint, quart, gallon, ounce, pound, etc., the experience of the pupil is an important factor in determining the order in which these should be taught. By a judicious selection of such units, much may be done towards connecting the school life of the pupil with his home life. Facility of concrete presentation and simplicity of relation to other units will also be considered in this connection.

6. Numerical Relations.

Number is the product of the action of the mind in defining a quantitative whole. The first step is one of analysis, whereby the whole is thought of as being made up of like parts. This gives rise to the idea of unit. It is a process of simplification on which numbering is conditioned. If the mind could go no farther than this there could be no numbering or numerical

measurement. The like parts must be related to one another and to the whole, in some degree at least, before any progress can be made in determining the whole quantity. The first efforts may, indeed, produce rather indefinite results, but they are the beginnings of greater things. The mind proceeds from the vague to the definite. In numbering, this has two applications, first, as to the units themselves, second, as to the manner in which the units are related.

It has already been hinted that there is a progressive order, from the vague to the definite, in attitude of the mind with reference to units as advancement is made in number. The units first employed by the child are regarded as alike, but in themselves they are quantitatively undefined by him. The relating of such units cannot rise higher than mere numbering. An aggregate composed of such can be defined in so far as the *number* of the units is concerned, but it cannot be defined as a whole quantity. For example, when we speak of 10 apples we think only of the number, the *how many*, we know nothing definite about the *how much*. If we wish to determine the whole quantity we must do so in terms not of one apple, but of some fixed standard of measurement, as one pound. We cannot say 3 apples : 15 apples :: 1 : 5, but we can say 3 lbs. of apples : 15 lbs. of apples :: 1 : 5. If we do make the former statement we have first to regard the units as quantitatively equal to one another, otherwise the statement is not necessarily true. Undefined units may be regarded as taken together and separated; they may also be enumerated, but a direct comparison of quantities based on such is impossible. Even in the case of measured standard units it is not until the exactness of a mathematical conception of them is gained, that it is possible for the mind to relate in the highest degree. This indicates not only the order in which units should be presented, but also the order in which they should be related.

From these considerations we see that there are two distinct stages in relating—

(i) Defining aggregates by numbering.—*Number.*

(ii) Defining aggregates by comparison.—*Ratio.*

As the evolution of number is nothing more or less than the evolution of thought in a certain direction, numerical relations must necessarily present different degrees of complexity corresponding to different stages of mental development. This can easily be shown to be the case. It is plain that units of any kind can be numbered. The best condition for numbering them, however, is when they are regarded as not possessing any quantitative value in themselves. This condition being fulfilled, the mind is impelled in the direction of numbering, as there is no other means of determining the aggregate. Further, comparison or ratio is conditioned on numbering. It would be impossible to say that one yard is a measure 36 times as great as one inch unless the one-inch units composing the aggregate represented by one yard could be numbered; but it would be equally impossible to make the statement without, in addition, taking cognizance of the measure of the unit itself. This makes the distinction between number and ratio perfectly clear, while it shows at the same time that number is an element of ratio.

There is no psychological necessity for selecting such arbitrary standard units as yard, second, gallon, etc., as any other units of definite measurement would do equally well for purposes of comparison. The point to be noticed is that the aggregate must be thought of as composed of units *which are themselves either absolutely or relatively defined as to quantity*, in order that such aggregate may be determined, not by numbering the units merely, but by relating them quantitatively to one another and to the whole. This indicates the educational value of *practical measurement* in a course of study.

The preceding statements may be illustrated thus:

(*a*) Suppose we take three different quantities of 5 pears each, the unit being undefined quantitatively. We may add them together, 5 pears + 5 pears + 5 pears = 15 pears, or we may go back and count the number of addends and say 5 pears × 3 = 15 pears. The latter expression cannot mean anything more than that if we combine 3 groups of pears, having 5 in each, we get a total of 15 pears. This is an example of numbering; the idea of ratio cannot be present as the groups of 5 pears may be all different from one another, the units being unrelated quantitatively. This represents the first and lowest form of determining an aggregate.

(*b*) Suppose we regard the groups to be equal quantitatively, but the individual units in each group to be unrelated. Then 5 pears + 5 pears + 5 pears = 15 pears has implied in it the idea of ratio, although that idea is not fully expressed until we put it into the form 5 pears × 3 = 15 pears. 3 here indicates the ratio between any one of the equal groups and the whole quantity. The thought in mind may be expressed either as 5 pears × 3 = 15 pears, or 5 pears = $\frac{1}{3}$ of 15 pears. But as the individual pears are unrelated quantitatively, we cannot say 1 pear is $\frac{1}{15}$ of the whole quantity of 15 pears, neither can we say 1 pear is $\frac{1}{5}$ of each group of 5.

(*c*) Suppose we regard the pears to be all equal to one another. Then the groups of five pears must be equal. Here the whole quantity or any part may be defined in terms of the unit 1 pear. In this the idea of ratio may be as completely in mind as if a recognized standard of measurement as pound, peck, etc., were used. We can say 10 pears = 1 pear × 10 = 2 pears × 5 = $\frac{2}{3}$ of 15 pears, etc.

(*d*) Suppose that there is only 1 pear. We can think of the repetition of that one as expressed by 1 pear × 15 = 15 pears,

Here the idea of ratio is necessarily present, and that in the highest sense as an abstract conception. It is impossible to think of 1 pear $\times$ 15 = 15 pears in this way without having the corresponding thought in mind, 1 pear = $\frac{1}{15}$ of 15 pears. Indeed, these two are but complementary phases of one thought.

These considerations are of immense importance as a guide in the preparation of a course of study, as well as in directing the thought of the pupil as he progresses from one stage to another. Although there are many gradations between the initial step of determining an aggregate vaguely by numbering units which are unrelated in every respect, but that they constitute the whole, and the highest process of relating, unit to unit or units, and unit or any number of units to the whole, three divisions may serve for all practical purposes in teaching, as indicating what should be emphasized during consecutive periods of school work. For want of better terms these may be designated the *Numbering*, *Measuring* and *Comparing* periods.

During the first or numbering period the chief aim of the teacher will be to direct the attention of the pupil to the *how many* of the aggregate. It must not be thought that a perfect knowledge of numbering can be gained at this early stage. Nothing of the kind is to be attempted, for it would be impossible. Both theory and practice have established the fact that not until the *how much* is understood can the *how many* bear its full meaning. The object is to acquire such power in determining aggregates numerically as will enable the pupil to take, easily and naturally, the next step along the line of progress. From the beginning, however, the end must be kept in view if the most direct course is to be taken. The idea of number as the *measure* of quantity must be made prominent. This suggests that more stress should be laid on 6 + 6, 8 + 8, 4 + 4 + 4, 8 + 8 + 8, than on 6 + 5, 8 + 7, 9 + 8, etc. These latter are not, of

course, to be neglected, but they are not to receive as much attention as the former when *numbering* is the main object. Lessons leading up to measurement and comparison will be given from the beginning. They are necessary, for the reason that the child has already begun the work of measuring and comparing. But he has done it in a very indefinite manner. The vague notions he has gained will be made the basis; also the same means he has already employed will now be used to greater advantage under proper direction, together with such more efficient means as his increasing power may warrant. This is the time for eye-measurement, foot-measurement, etc. At no later date will he so well learn to use the instruments bestowed by nature, for he is now completely dependent on them. The motto throughout this part of the course should be, *secure power in numbering by means of undefined units.* Addition and subtraction will be practiced to a greater extent than multiplication and division.

The pupil having gained a fair command of number, is ready at the commencement of the second or measuring period to give his attention to the quantitative value of units, and thus become prepared for the more difficult task of comparing quantities on a well-defined basis.

It has been shown that the units making up an aggregate must be quantitatively related to one another, otherwise the idea of ratio cannot be present.* Such being the case the problem for the teacher is this: How can I best direct the thought of the pupil so that he will consider the quantitative relation of units? The answer is plain: Bring before the notice of the pupil such aggregates as are measured by fixed standard units as foot, yard, peck, pound, cent, etc., and give

* It may be contended that as the process of relating is a purely mental one, it cannot matter as to whether the units are actually equal to one another or not. Quite true, the whole question is one of mental attitude. But how is the proper mental attitude to be secured?

him plenty of practice in measuring them. At first the pupil will measure 10 feet with a foot rule as a mere numbering process, just in the same way that he would count 10 apples, but in measuring the distance 10 feet his attention is constantly directed to the length of the foot rule. This work will delight the pupil as new activities are now called into play. He sees more in number than he did before. He feels that he is growing in strength. The work in numbering will be continued with greater zest than ever. This is the time to fill in what was previously dealt with in outline. The motto here should be, *secure power in numbering by means of measured units.* Multiplication and division will become more and more prominent during this period, but much practice will still be given in addition and subtraction. Factors and multiples will also be dealt with to some extent.

The pupil having gained considerable mastery over the difficulties of numbering measured units, is in a position to relate the units to one another, and thus determine the *how much* of the aggregate or of any part of it. In fact, the pupil began to relate units to one another as soon as his attention was directed to their quantity. He is now to be brought into a full consciousness of the higher thought processes of numerical ratio. The motto here is, *secure power in comparing quantities.* Fractions, reduction of denominate numbers, etc., belong to this period.

7. Numbering by Ones.

The combining of like objects into small groups and the separating of groups into their constituents are the external operations whereby children acquire their first notions of number.* The presence of objects is not enough. It is only when there is consciousness of a whole as being made up of like parts that progress in number is possible. This condition being fulfilled, the mind naturally reacts on the material

* Strictly speaking, the order here given should be reversed. See pp. 2, 4 and 5.

objects presented to it, and takes cognizance of its own acts of attention in passing from the whole to the parts (units), and from the parts (units) to the whole. These acts of attention are related to one another, and number is the result. Single objects, as apples, sticks, etc., which are quantitatively undetermined, when regarded as units, are the least definite possible. Therefore, counting single objects is the fundamental process in numbering.

In counting, two distinct mental operations are involved:

(i) Separating and combining units. Although separating units and combining units are apparently different, they are in reality only the analytic and synthetic phases of one mental act. This is the mode of thought which finds its expression in the arithmetical processes of subtraction and addition.

(ii) Numbering units. The enumeration of units is the second mental act in counting. Although it is conditioned on the separating and combining of units, it is clearly a different mental process. It consists, essentially, in assigning to each unit its place in the whole series of units, giving rise to the use of such terms as first, second, third, etc. Until the pupil becomes conscious of a first, second, third and fourth act of attention in counting four objects, say, it is quite impossible for him to number the objects. As soon as he establishes the order of the different acts in the series and synthesizes them, he has found the number, whether he can give that number a name or not. The arithmetical processes of division and multiplication give expression to this mode of thought.*

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*Multiplication represents two distinct stages in the evolution of number:

(i) The exact quantitative value of the unit is not considered. Here multiplication implies *finding the sum* of addends numerically equal to one another and *numbering* them. In this sense multiplication is nothing more than a short form of addition.

(ii) The multiplicand is regarded as a unit to which the product bears a fixed ratio, such ratio being defined by the multiplier.

Similarly division also represents two distinct stages of thought.

These important aspects of multiplication and division should be kept in mind by the teacher.

From these considerations it is easily inferred that counting forms the basis of the four simple rules in arithmetic.*

8. Numbering by Groups.

The child naturally counts at first by ones, but as the labour of counting increases with the number of units, a point is soon reached when he finds it advantageous to increase the unit. As soon as this is the case he counts by twos, threes, etc., with renewed pleasure. Why? Because he has found out a new and more definite method of determining the aggregate. Why more definite? Simply because the new unit has a quantitative value of its own. True, it is not well defined, but sufficiently so to require greater effort than before in numbering the whole quantity by means of it.

During this early stage the pupil is quite satisfied if he can number the objects before him, using first one unit, then another, etc. Generally speaking, he is not concerned with a minute analysis of all the possible combinations and separations of the number, for the reason that such analysis neither supplies any want which is felt, nor does it give consciousness of increased power. Every primary teacher knows how wearisome drill on combinations and separations becomes to the pupil, until he attains to a certain degree of familiarity with number. The cause of the whole difficulty is that an attempt is made to compel the pupil to memorize facts which are not mastered. But let a pupil find out that he can number 12 objects by twos, threes or fours instead of ones, and he feels at once the importance of his discovery. He has found out a new method of doing something he has a desire to do. How different is the pupil's attitude towards the study of number when he is required to deal with all the facts of the number *seven* before he is allowed even to utter *eight!*

* For an account of the genesis of the fundamental operations, see *Educational Review* for January, 1893, pp. 46 and 47.

The pupil should be given practice in forming groups and in recognizing readily groups which he has counted. The purpose of this is to make him so familiar with whole groups that he will detect them readily as units in the numbering of larger wholes. The pupil should be put in a position to discover a unit by which he can number an aggregate, rather than build up from the unit.* Aggregates which are easily related to units well known to the pupil should be selected. For example, if the pupil knows 3, 6, and 9 so well that he can use them readily as units, he may be asked to measure 18 in as many ways as he can. It would be a mistake to ask him to measure 18 by 6's, as naming the unit would deprive the pupil of the pleasure of gaining a new victory for himself. The object should be to get the pupil to grasp the idea of numbering. The remembering of results at the beginning is of little importance, except in so far as it carries with it added power.

Counting and grouping is work that is well adapted to pupils who enter school at the age of 5 or 6 years. It is a serious error to attempt much work in number with young children, as they have not the power necessary to deal with it. Regular, systematically arranged lessons on number may well be deferred until the pupil is at least seven years old. Up to that time the subject should be dealt with incidentally. It must not be forgotten that, no matter how clear the presentation on the part of the teacher may be, numbering—not the mere handling of objects—makes no small demand on the part of the pupil.

9. The Use of Objects.

The extent to which objects should be employed in teaching number will depend on the mental condition of the pupil. In the case of young children, speaking generally, objects should be used freely during the early part of the school course.

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* See definition of the term *unit*—page 6.

The primary notions of number are occasioned by experience with material objects. Such early perceptions are very vague, indeed. For example, a child will be able to distinguish three objects from two long before he has a true conception of the numbers two or three. These fundamental, though imperfect, ideas being based on sense perception, it is the duty of the teacher to present material of such a kind and in such a manner as to awaken the higher thought activities of which number—the relation between a unit and a whole quantity—is the product.

Again, the power to form direct perceptions is very limited, indeed; therefore, any advancement beyond the very lowest numbers must be made by relating one number to another. This consideration furnishes another reason why the perception of relations by the pupil should be the aim of objective teaching from the beginning.

Objects, as employed in teaching number, serve three purposes:

(i) They are a means of occasioning mental action.*

(ii) They reveal to the pupil what he has to do mentally, or what his mind is actually doing.

(iii) They furnish a test by which the thought activity of the pupil may be determined. The pupil may make a statement of a fact which in reality means little or nothing to him. This will be shown at once by his failure to illustrate it by means of objects or objective representation.

The foregoing applies to all forms of objective teaching, including representation by lines or diagrams.

* In actual practice it is easy to fall into the error of assuming that the performance of operations with objects is *necessarily* accompanied by that form of thought activity which results in numbering them. A moment's reflection will show, however, that the attention may be fixed to such a degree upon qualities—form, color, texture, etc.—which characterize the objects individually, as to prevent the very kind of mental action which their presence is intended to stimulate.

When the pupil can think readily of number as apart from particular objects, their use can serve no further purpose in developing the notion of the *how many*, and in so far as this purpose is concerned the use of objects should be discontinued except as a test of the pupil's work. In the case of young children, however, it is better to have too much objective teaching than too little; in other words, it is better to fail in reaching the extreme limit of the pupil's power than to appeal to something which has not yet been developed. Nevertheless, it must always be regarded as a defect in teaching to allow the pupil to employ objects if it is possible for him to think out numerical relations without their aid. Here is a case in point: A pupil fails to find the sum of 9 and 6; he is immediately sent to the number table to find the answer by counting blocks. If such a course is necessary, either the teaching is wofully bad or the pupil lacks something which no teaching can supply. If number is essentially a *thought process*, then the pupil should *relate* $9 + 6$ to some known combination, say $10 + 6$, $6 + 6$, or $9 + 3$. Such an analytic and synthetic operation will develop power to deal with new difficulties, while the mere handling of the objects can do little more than inform the pupil of the required result.

A clear distinction must be made between the use of objects in gaining a proper idea of number—the *how many*—and their use in presenting standard units of measurement, illustrating numerical relations, etc. In the latter sense, objective teaching should have a somewhat prominent place throughout the whole public school course in arithmetic.

The reproductive imagination should be frequently appealed to in arithmetical work. For example, in the exercise, *Find how often 4 inches is contained in 6 feet*, the pupil may form a mental picture of a line instead of representing it by means of a diagram. The value of such exercises does not seem to be fully appreciated.

10. The Fundamental Rules.

All numerical calculation, whether elementary or advanced, is based on the fundamental processes of addition, subtraction, multiplication and division. A perfect command of these is as necessary for the arithmetician as is familiarity with the keyboard for the pianist. No musical performer could hope for success in interpretation or expression so long as he is hampered by lack of acquaintance with the instrument he uses. In like manner the arithmetician need not expect to attain to a high degree of perfection in his art until he has first mastered the fundamental processes. A theoretical knowledge of them is not sufficient; many a one understands them and can explain them who is not able to calculate either accurately or rapidly.

There are several reasons why this part of arithmetic should receive careful attention.

(i) *It develops the power of attention.* The pupil is required to concentrate his thought fully on what he is doing in order to make progress. The work is of such a nature that any interruption or lack of effort is at once detected.

(ii) *It tends to form the habit of being accurate.* Arithmetic, above all other elementary studies, should secure exactness in thinking and doing.

The teacher who insists on strict accuracy in all school work does a great deal more for the pupil than to teach him the subjects indicated by the programme of studies. The pupil is getting a preparation for actual life, where accuracy is a matter of the highest importance.

(iii) *It tends to form the habit of thinking quickly.* In performing the operations of the simple rules, the attention shifts rapidly from one object of thought to the next, producing

a continuous mental current, the course of which is modified by every new act of attention.

(iv) *Familiarity with the fundamental rules is necessary for the business of life.* Probably nine-tenths of the arithmetic practised outside of educational institutions consists of simple addition and multiplication, and probably ninety-nine-hundredths of it does not go beyond easy applications of the four simple rules. Where higher work is required, as in the measurement of timber, the computation of interest, etc., tables are used, so that even in such cases a good knowledge of the fundamental rules will generally suffice.

While these facts do not form an argument in favour of confining the study of arithmetic within narrow limits, they show how important it is to teach every pupil to add, subtract, multiply and divide numbers accurately and readily. If the pupils that leave our schools cannot perform well these every day operations of arithmetic, surely something is wrong. It is feared that this is too often the case.

How are accuracy and speed in calculation to be secured? By careful teaching from the beginning, and by constant and well-directed practice throughout the whole public school course.

Accuracy should be first aimed at. Why is it that a business man succeeds in training a boy to be exact, while the teacher so often fails? Simply because the former impresses the boy with the importance of strict accuracy, while the latter does not. This shows the educational value of *doing work under a proper sense of responsibility.* Blundering in school, just as in actual life, should be regarded as a serious matter, and until such is the case, little improvement need be looked for.

Accuracy being secured, rapidity will come only through

plenty of careful practice. The pupil should be given sufficient time for an exercise, but no more. As he gains in speed, of course, the time allowed will be correspondingly diminished, so that he will always be required to make his best effort.

11. Arithmetical Processes.

The question as to when the rationale of arithmetical processes should be dealt with is one on which there is much difference of opinion among the very best teachers. Some contend that it is a waste of time and a useless expenditure of mental energy to enter into an explanation of such numerical processes as adding, subtracting, multiplying, dividing, find the G.C.M. (formal), extracting the square root, etc., at the time they are first taught. Others maintain that the pupil, as an intelligent being, should not be required to perform an operation that he cannot explain, or at least understand. Those who take the former position generally defend it on the ground that arithmetical processes, in their relation to the whole subject, are merely a means to an end, and as such it is quite unnecessary to enter into any explanation of them so long as the desired end can be attained without it.

The main questions to be considered are: (*a*) At what time is the pupil able to put forth the mental effort necessary to investigate the principles underlying a process? (*b*) To what extent does a knowledge of the principles on which a process is based affect the performance of it? (*c*) What is the exact place in the development of the whole subject at which the reasons for a particular process must be understood?

Some arithmetical processes are much more easily understood than others. Compare, for example, those of addition and subtraction. To the pupil who can combine and separate numbers readily, and who is familiar with the decimal system of notation, "carrying" in addition will present almost no

difficulty, while "borrowing" in subtraction will give no end of trouble if the explanation of it is pushed beyond the very simplest cases. Is the pupil not to be allowed to use the process of subtraction as an instrument of thought simply because he cannot fully comprehend it? By all means he should be taught *how* to perform the operation so that he may be able to use it. As *thought processes*, addition and subtraction are complementary, and they should therefore go together. This is the controlling fact to which the teaching of the formal processes must be subordinated.

The following should be carefully distinguished from one another:

(i) The *process*—the instrument by means of which a certain work is done.

(ii) The *use* of the process—the work that the instrument is capable of doing.

(iii) The *rationale* of the process—the construction of the instrument.

These may be illustrated as follows: Suppose a pupil to have such a knowledge of factors and common factors as will enable him to find for himself the H. C. F. of two or more small numbers, as 60, 84, 112. He resolves the numbers into their prime factors thus:

$$60 = 2 \times 2 \times 3 \times 5.$$

$$84 = 2 \times 2 \times 3 \times 7.$$

$$112 = 2 \times 2 \times 2 \times 2 \times 7.$$

After examining the prime factors of the numbers, he discovers that 2×2 is the H. C. F. The whole process is clearly understood by the pupil from beginning to end, for he has

worked it out himself. He is then asked to find the H. C. F. of 1908 and 2736. He proceeds thus:

$$1908 = 2 \times 2 \times 3 \times 3 \times 53.$$

$$2736 = 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 19.$$

$2 \times 2 \times 3 \times 3$ is found to be the H. C. F. of the given numbers.

The pupil is now required to scrutinize carefully and to state clearly what he has done in finding the H. C. F. required. He finds that he has retained all the common factors of the numbers, and eliminated all the factors which are not common. The teacher then shows him a process—places an instrument into his hands—by means of which the common factors may be retained, and the factors which are not common may be got rid of more expeditiously than by the method formerly employed thus:

$$\begin{array}{r|l}
2)\,1908 & 2736(1 \\
1656 & 1908 \\
\hline
3)\;\;252 & \;\;828(3 \\
216 & \;\;756 \\
\hline
36 & \;\;\;\;72(2 \\
 & \;\;\;\;72 \\
\cline{2-2}
\end{array}$$

The pupil understands fully the use of this new process—the work the instrument is capable of doing. He can, therefore, employ it intelligently. But he may not be able to comprehend the fundamental principles underlying the process, much less explain them.

This form may be given, but it affords no explanation, it

merely shows clearly how the formal process of finding the H. C. F. serves the purpose for which it is intended.

$$
\begin{array}{r|l}
2)\,2\times2\times3\times3\times53 & 2\times2\times3\times3\times76\,(1 \\
2\times2\times3\times3\times46 & 2\times2\times3\times3\times53 \\
\hline
3)\,2\times2\times3\times3\times7 & 2\times2\times3\times3\times23\,(3 \\
2\times2\times2\times3\times6 & 2\times2\times3\times3\times21 \\
\hline
2\times2\times3\times3\times1 & 2\times2\times3\times3\times2\ (2 \\
 & 2\times2\times3\times3\times2 \\
\cline{2-2}
\end{array}
$$

The points to be noted in teaching arithmetical processes are:

(i) The pupil must feel the need of the process before it is taught. This means that he has done all he can do conveniently without it, and that he is fully cognizant of its purpose. Give the pupil plenty of opportunity to reason from first principles.

(ii) The process is presented by the teacher and the pupil is immediately required to apply it.

(iii) The process is examined when the pupil is able to deal with it, so that he may discover the principles on which it is based. To teach process, and reason for process together, is, generally speaking, a pedagogical error, because two difficulties have to be overcome at once, and further, because it is only after a pupil has become familiar with a process that he can be in the best position to investigate it.

12. Problems.

During every stage of progress the pupil should be given ample opportunity of applying his knowledge practically. Problems should therefore have a prominent place throughout the whole course in number.

The greatest care should be exercised by the teacher in selecting or making problems so that they may be well

adapted to the requirements of the pupil. To give problems which present no stimulus is to waste time. This surely finds its application in the thousand and one problems which are so often given on the number 5. On the other hand, to ask a pupil to solve a problem requiring for its solution a process of thought higher than that to which he has attained, is to attempt the impossible. There is but one error greater than this one, and that is for the teacher to give an explanation of the problem and then try to "sense" the pupil into understanding it. Unfortunately, too many pupils have been subjected to such unprofessional treatment, and have had to suffer its evil consequences.

Viewed from the standpoint of discipline, a problem is worth but little to a pupil unless he solves it himself. As to whether the solution is the neatest or best matters but little at first, provided that it is original. After the pupil has discovered a path for himself, however, his attention may be directed to it in order that he may find out whether or not he might have chosen a better one. Short cuts taken at the proper time form a valuable means of discipline, as they necessitate vigorous thought. For example, the solution of the problem, *I sold a horse for \$75 gaining 25 %, what % should I have gained if I had sold him for \$80?* may take several forms as,

(i)
$$\begin{aligned} \tfrac{125}{100} \text{ of cost} &= \$75. \\ \therefore \tfrac{1}{100} \text{ of cost} &= \tfrac{1}{125} \text{ of } \$75. \\ \therefore \tfrac{100}{100} \text{ of cost} &= \tfrac{100}{125} \text{ of } \$75. \\ &= \$60. \end{aligned}$$

Therefore gain in second case is $\$80 - \$60 = \$20$.

$$\begin{aligned} \text{The gain on } \$60 &= \$20. \\ \therefore \text{The gain on } \$1 &= \tfrac{1}{60} \text{ of } \$20. \\ \therefore \text{The gain on } \$100 &= \tfrac{100}{60} \text{ of } \$20. \\ &= \$33\tfrac{1}{3}. \end{aligned}$$

The gain % in second case is $33\frac{1}{3}$.

(ii) $\$75 = 125\ \%$ of cost.

$\therefore \$1 = \frac{1}{75}$ of $125\ \%$ of cost.

$\therefore \$80 = \frac{80}{75}$ of $125\ \%$ of cost.

$= 133\frac{1}{3}\ \%$ of cost.

The gain % is therefore in second case $33\frac{1}{3}$.

(iii) $\$75 = 125\ \%$ of cost.

$\therefore \$80 = \frac{80}{75}$ of $125\ \%$ of cost.

$= 133\frac{1}{3}\ \%$ of cost.

The gain % therefore in second case is $33\frac{1}{3}$.

(iv) $\$75 = \frac{5}{4}$ of cost.

$\therefore \$80 = \frac{80}{75}$ of $\frac{5}{4}$ of cost.

$= \frac{4}{3}$ of cost.

The gain in second case is $\frac{1}{3}$ of cost $= 33\frac{1}{3}\ \%$.

Suppose that the first form of solution is the one which is obtained. It is roundabout, and therefore indicates a degree of weakness on the part of the pupil, but if it is his own, it has developed power. The pupil may be asked to attempt a shorter form of solution. A question to put him on the track may be given, but nothing more. If he gets the second form, he shows a gain in relating, although there is still an indication of weakness in that he has not made a direct comparison between $75 and $80. The third and fourth forms here given are preferable to the others, in so far as they indicate a greater degree of ability in establishing a ratio between quantities. No form should be placed before the pupil in any case if the purpose is to secure discipline, and not merely to show how to solve the problem. To give a form of solution before the pupil has done the thinking corresponding to it is a delusion. To relate all quantities to *unity* when more difficult ratios can be established, is another source of weakness.

What are the characteristics of an arithmetical problem?

(i) Certain conditions are given.

(ii) A quantity is to be determined.

(iii) Definite relations are expressed connecting the undetermined quantity with the given conditions.

The difficulties which the pupil will meet in problem work are these:

(i) Comprehending the data.

(ii) Analyzing the given conditions and determining the relations they bear to the unknown quantity.

(iii) Deciding upon the numerical operations that will correspond to and carry into effect the thought process of solving the problem.

(iv) Performing the numerical operations necessary to evaluate the required result.

These it may be well to consider somewhat in detail.

(i) *The conditions given may not be fully comprehended.* If the data are not understood, the pupil cannot even begin to do what is required of him. Indeed, the difficulty may not be an arithmetical one at all. Compare, for example, the problems, *Find the cost of 10 desks at $4.87 each*, and *Find the cost of a bill of exchange for £10 at 4.87.* The former of these is suitable for almost a beginner, the latter might put some teachers to the test, and yet the arithmetical operations required for their solution are identical. The duty of the teacher is to lead the pupil by means of questions or illustrations to grasp the con-

ditions given. If they are beyond his reach, the problem is unsuitable, and should be withdrawn.

The second of the foregoing problems indicates clearly that quantity relation is not the only thing to be dealt with in what is usually termed arithmetic. Often the greatest difficulty which the pupil has to contend with arises from the qualitative elements of the data. He lacks the experience necessary to enable him to interpret the given conditions and to represent them to himself in such a manner as to be able to apply either his mental power or his knowledge of number in relating them. This shows the disadvantage of attempting to teach interest, discount, exchange stocks, etc., to pupils, before they have a proper conception of the business transactions on which such divisions of the subject are based.

(ii) The pupil may not be able to analyze the problem sufficiently to *locate the difficulty definitely.* In that case he does not know the exact point at which to concentrate his energies. For example, a pupil fails to solve, *Find the number of cords of wood in a rectangular pile which is 7 feet high, and which covers $\frac{1}{8}$ of an acre.* He knows what $\frac{1}{8}$ of an acre means, and he can calculate the number of cords when the length, breadth and height are given. If asked why he cannot get the answer, he will probably say that the length and breadth are not given, but that is not the real difficulty at all. It is this, he does not see the connection between the linear measurements of length and breadth in one case, and the surface measurement in the other. The duty of the teacher is to question the pupil, so as to lead him up to the exact point of difficulty, and then leave him to grapple with it as best he can. To solve the problem and then try to get the pupil to see through it would be a serious pedagogical error.

It is not to be inferred from what has been said that no value whatever attaches to following step by step a neat, logi-

cally arranged solution worked out by another; for, under certain circumstances such an exercise may prove a very efficient means of stimulating independent effort. This is a very different thing from giving assistance whenever a difficulty is met. We must not forget that self-reliance, determination, and strength are developed by overcoming difficulties and surmounting obstacles, and in no other way. The instructor who keeps this clearly in view will do much more for the pupil than teach him how to solve arithmetical problems.

Again the difficulty may be that of *determining the connection between what is given and what is to be found.* We may take, by way of illustration, the following problem: *The amount of $100, bearing compound interest for two years, payable annually, is $121. Find the rate per cent.* The pupil can find the compound interest on a sum of money for a given time, but he fails to solve the foregoing. What is to be done? Surely not to give him a form of solution and ask him to think it out, or to lead him by a series of questions*—perhaps logically arranged, but arranged by the teacher, not by the pupil—to arrive at the answer. Where is the real defect? It lies in the fact that the pupil did not secure a proper grasp of compound interest. He looked at the matter from one point of view only. What should be done is this: He should be asked to perform the operation of finding the compound interest over again, and to retrace the steps taken in finding the answer. If he is able to deduce from his work the relation $A = P(1 + r)^2$, he is in a position to solve the problem; if he has not power to do this, it is too difficult for him.†

(iii) It is unnecessary to illustrate the case where the pupil has difficulty in determining the operations to be performed. We frequently hear such questions as this asked by beginners,

*Proper questioning at the right time is a good thing, but the kind of questioning so often practised, which is merely an interrogative form of telling, is pernicious.

†See page 104.

whether shall I multiply or divide? Such is the result of defective teaching. The important fact that number is the product of mental action has been lost sight of. The pupil has been shown how to perform operations without learning how to make use of them. The remedy consists in getting him to apply whatever knowledge he may have, and to illustrate his work objectively when necessary.

(iv) Lack of ability to calculate accurately and readily is not by any means uncommon. The weak points should be noted by the teacher, and suitable, well arranged exercises on the simple rules should be given. This matter is referred to elsewhere.

One other difficulty may be worthy of consideration. There may be *known facts bearing on the solution of the problem which the pupil cannot summon to his aid.* Sometimes a mere suggestion may serve to lead him back to a related known fact or principle. For example, a pupil is asked to find the area of the right-angled triangle whose sides are 6 feet, 8 feet and 10 feet. He draws a diagram of it, and he attempts to find the required result, but he cannot succeed. He then informs the teacher that he does not know how to find the area of the triangle. The teacher asks him to say what he knows about finding areas. Probably this may be quite sufficient to enable the pupil to relate the area of the right-angled triangle to that of the corresponding rectangle.

The recitation may take various forms, according to the object in view:

1. *Stating in simple language the conditions of the problem.* This is often a very necessary preliminary exercise. It awakens interest and it assures the pupil that he is in possession of the facts on which the solution is to be based.

Confidence will thus be gained, and the pupil will engage in the work with a determination to succeed.

2. *Analyzing the conditions carefully and relating them to the required answer.* This is what may be termed *solving* the problem. In the case of advanced classes it is generally better to make no use of figures at all, except in so far as they may be required to express relations. The object is to deal with the main thought process of the problem.

3. *Stating the operations to be performed in arriving at the answer.* This will be found to be an excellent exercise after the pupil has had sufficient time to think over the problem. It directs attention to the uses of the fundamental rules as instruments for serving the purposes of thought. In addition to this, it is a time-saving exercise. Probably ten problems can be dealt with in such a manner in the time taken by two or three worked out from beginning to end. At the close of the recitation, or some time afterwards, the pupils should be required to perform the operations so as to find the required result, as a quantity in its simplest form. The latter should often be an oral exercise.

4. *Thinking out the solution, finding the answer, and stating the whole process in written form.* This is an excellent exercise, especially when large numbers are involved. It gives the pupil an opportunity of doing the whole work and of expressing it clearly and neatly. The time given must be limited, however, so that the pupil will not fall into sluggish habits. This exercise, though valuable, should not be allowed to usurp the place properly belonging to other forms.

5. *Thinking out the solution, finding the answer and stating it orally.* This invaluable exercise should have a prominent place in every school-room in which number is taught. Mental arithmetic is highly advantageous for several reasons: (i) It

requires the pupil to hold all the conditions and relations of the problem in mind at one time, thus tending to give him a strong mental grasp. (ii) It leads to accurate and rapid calculation. (iii) It is the most practical in everyday life. (iv) It stimulates the pupil to put forth his best effort—an education in itself.

There is far too much use made of the pencil in our schools at the present time. Twenty minutes devoted to vigorous mental work is worth more than an hour spent in leisurely performing written exercises.

New principles should always be developed by means of oral exercises, using the very simplest numbers. In fact, the greater part of the whole time devoted to arithmetic should be taken up with mental work. Improvement in this direction is greatly needed.

13. Educational Value of Problems.

The value of problems as a means of intellectual training is so generally recognized that it is unnecessary to discuss it here at any great length. But the consideration of two or three points which are likely to escape notice may not be out of place.

Number, in common with other school subjects, has its mechanical side. Fundamental facts must be remembered, processes must be learned, or there cannot be progress. In this there is always a danger that the pupil may become so confirmed in a habit of thought belonging to a lower stage of mental activity, as to prevent him from passing easily and naturally to a higher. Herein consist chiefly the evil effects of memorizing facts before they are comprehended, of learning tables which are not understood, of dealing with fractions which are not fractions at all (to the pupil), of becoming familiar with forms of expression which do not correspond to thought

activity, etc. The solving of problems is one of the best safeguards against this, as the pupil is constantly required to relate new ideas to those already in his possession. By this means he is enabled to rise by easy gradations from one stage of advancement to the next, always making use of what he already knows. Two conditions are here assumed :

(i) That the problems are suitably arranged in a progressive series.

(ii) That in solving them the pupil does the whole work for himself.

But if these conditions be not fulfilled, problems may be a source of injury rather than benefit. By way of illustration take the problem, *If 5 lbs. of sugar cost 30c., what is the cost of 10 lbs.?* Supposing the pupil not to have grasped the idea of ratio, the problem is clearly beyond him. Although it appears simple, the knowledge which he has cannot be applied in solving it. But he may be shown how to get the answer thus :

The cost of 5 lbs. is 30c.
$\therefore$ " 1 lb. " $\frac{1}{5}$ of 30c.
$\therefore$ " 10 lbs. " 10 times $\frac{1}{5}$ of 30c. = 60c.

Keeping in mind the assumption made, it is at once seen that the statement, the cost of 1 lb. is $\frac{1}{5}$ of 30c. cannot be the expression of the pupil's thought. He may admit the truth of the statement, but that is neither here nor there, because he has not thought it out. For all this, experience proves that a few repetitions will enable him to make use of the form in the case of other similar problems. But if the problem be varied thus : *If 5 men can do a work in 30 days, in what time can 10 men do it?* he is at as great a loss as ever. Why? Simply because the solution given means nothing more to him than a

matter of form. It is indeed bad enough to repeat over and over again hundreds of times, such combinations $3+4$, $5+6$, etc., before they are understood, but it is infinitely worse to deliberately furnish the means for causing what ought to be a highly valuable intellectual exercise to degenerate into a mere mechanical operation, and thus arrest mental growth perhaps for many a day to come.

Next, let us suppose that the problem is presented at the time the pupil is prepared for it, and that he is left to his own resources for its solution. The chances are ten to one that he will not adopt the standard school form of *three* lines. He will probably compare the quantities of sugar, and discover that there is twice as much in the second case as in the first. He will connect this idea with that of the given price, and immediately arrive at the answer. But it matters not as to the form of solution, whether it should be the best or otherwise. The point is this: *the pupil has solved the problem independently, and he has gained strength in doing so.* His power to grapple with new difficulties has been increased. He is stimulated to greater effort, having experienced the pleasure of gaining a victory.

Another advantage of problem work is that it reveals defects in teaching. Take the following problem: *If .3 of a farm cost \$2400, what will $\frac{3}{5}$ of it cost?*

Here is the solution actually given by a student doing somewhat advanced work:

$\frac{3}{10}$ of the farm cost \$2400.			
$\frac{1}{10}$	"	"	$\frac{1}{3}$ of \$2400.
$\frac{10}{10}$	"	"	10 times $\frac{1}{3}$ of \$2400.
$\frac{5}{5}$	"	"	10 times $\frac{1}{3}$ of \$2400.
$\frac{1}{5}$	"	"	$\frac{1}{5}$ of 10 times $\frac{1}{3}$ of \$2400.
$\frac{3}{5}$	"	"	3 times $\frac{1}{5}$ of 10 times $\frac{1}{3}$ of \$2400 = \$4800.

The whole solution is made up of six statements, the second, third, fourth and fifth of which are unnecessary, or should be so at least. This form of solution may be attributed to one or more of the following:

(i) The relation between common and decimal fractions had not been properly taught, otherwise .3 and $\frac{3}{5}$ would have been related directly as .3 and .6.

(ii) The work of comparing quantities had not received due attention.

(iii) Attention had not been sufficiently directed to the advantage of taking short cuts. (The pupil should be encouraged in finding these out for himself.)

The information given by such a problem as this will greatly assist the teacher in determining the needs of the pupil.

A third advantage of problems consists in the fact that they afford the best opportunity, not only for applying new knowledge as it is acquired, but also for reviewing the old in relation to the new. Much time may be saved by giving this matter the attention it deserves. Tables of weights and measures, reduction, the compound rules, fractions, etc., may be kept fresh in the mind by means of problems which are carefully prepared with that end in view.

Problem work may be made an important means of connecting school exercises with home experience. A boy brought up on a farm will take great pleasure in solving such problems as: (i) A farmer tests his seed wheat and finds that 85 out of every 100 grains will grow. How much seed must he provide for 100 acres, allowing $1\frac{3}{4}$ bushels of good wheat per acre? (ii) A team travels at the rate of $1\frac{1}{2}$ mile per hour in ploughing a field of 25 acres. How many days of 10 hours each will be required for the work, supposing the width of the furrow to be 15 inches? (iii) Find the cost of 20 boards 16

feet long, 14 inches wide and 1 inch thick at $22.50 per thousand. (iv) A machine saws 40 cords of wood per day. How many days' work does a rectangular pile 80 feet long, 20 feet wide and $6\frac{1}{2}$ feet high represent?

Problems on marketing grain, purchasing goods, laying out and fencing fields and gardens, building barns, houses, etc., will be found particularly suitable for pupils of rural schools.

Such problems as the foregoing may be easily defended on the ground of their practical utility. But this is not the only, or the strongest argument in their favour. The data of these problems, being closely associated with the everyday life of the boy, direct his attention to his surroundings, and lead him to take a deeper interest in them. He thus gains a truer conception of the world he lives in, and consequently, of his duty in relation to it.

Much of the material employed in problems should be drawn from the other subjects of study, as geography, botany, physics, chemistry, etc. By this means the pupil's knowledge will gradually tend to become unified into a complete whole.

Problems should be thoroughly practical. Power to think out relations is only one of the ends to be kept in view. Another important end is *to train the pupil to do something useful.* It must not be forgotten that the value of power depends altogether on how it is applied. Hundreds fail in life from a lack of training in *doing* for every one who fails merely for want of mental power. The aim should be to secure discipline as the result of doing what is in itself useful.

14. Expression.

Language is the expression of thought. Number is the product of mental action. Numbering requires, in common with all other forms of thought, language which appro-

priately expresses it. In order to adapt the means of expression to the thought to be expressed and to its purposes, the introduction of numerical symbols became necessary. This gave rise to the use of characters—letters and figures. Several systems of notation have been devised but only two of them are now generally known, viz., the Roman and Arabic systems. In the expression of numbers, four things have to be taught:

(i) Spoken words.

(ii) Figures—Common notation.

(iii) Letters—Roman notation.

(iv) Symbols of operation.

Owing to the interdependence and interaction of thought and language, it is imperative that the pupil, in the lowest as well as in the highest stage of progress, should be in a position to express his thought clearly and fully. Applying this general truth to the case of number, it is seen that the pupil must be able, from the beginning, to use either words, figures or letters in order to make advancement in the subject. Now the practical questions to be considered are, at what time and in what order should these different forms of expression be taught?

Spoken words should be used from the beginning of the course. As the pupil becomes able to distinguish groups of two, three, four, etc., objects from one another he should be given the symbols one, two, three, four, etc., as spoken sounds. These will serve for him every purpose of thought and expression for some time. Indeed, it is quite possible to make considerable progress in numbering without a knowledge of any other symbols.

Figures may be taught as soon as the pupil has reached a true conception of number, that is, when he can think of a

whole quantity in relation to a unit. It must be kept in mind, however, that many vague determinations of quantities will necessarily precede the condition of thought here indicated.

When a new symbol is introduced the pupil should be required to use it frequently until he can associate it readily with the idea for which it stands. In such exercises as these the teacher will place the figure carefully each time on the blackboard, using it instead of the spoken word. Take up 6 blocks. Bring me 6, the teacher pointing to the objects. How many ones in 6? How many more than 5 is 6? How many threes in 6? How many twos in 6? etc. It is highly important that the symbol be of as accurate form as possible. The primary teacher, above all others, should write accurately and neatly on the blackboard.

The signs of operation should be presented one at a time as they are required for written exercises. As in the case of figures, they should take the place of words which are well understood by the pupil.

Roman characters may be introduced gradually after figures and symbols of operation are well known.

The decimal system of notation will present no difficulty until the number 10 is reached. It is best to show the pupil how to express 10, 11, 12, 13, 14, without any reference to the notation for these reasons:

(i) No difficulty is experienced in doing so. The pupil is perfectly satisfied to know the *how*, the *why* does not concern him.

(ii) By the time that, say, 14 is reached the pupil has gained some experience in relating these numbers to 10. He has a few facts at his command, and he is, therefore, in a position to make comparisons, $11=10+1$, $12=10+2$, $13=10+3$, etc.,

from these he immediately gets the idea of 1 ten + one, 1 ten + 2 ones, 1 ten + 3 ones, etc. The main thing is, of course, to make sure that the pupil has the thought giving rise to these forms of expression.

Plenty of practice should be given in both reading numbers and writing them after the system of notation is understood. The pupil should become thoroughly familiar with it as a preparation for performing numerical operations by means of figures.

Much attention should be given to expression in arithmetic. In fact the subject cannot be well taught without this. It demands logical arrangement and continuity of thought, exactness in the use of terms and clearness of statement. This is too often lost sight of in actual practice.

15. Course of Study.

OUTLINE OF WORK FOR FIRST PERIOD.

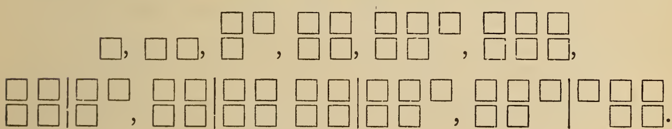
NUMBERING.

A—Counting—from 1 to 20.

B—Grouping—

(i) Objects—from 1 to 10.

(ii) Squares of equal size systematically arranged—leading up to comparison, as :



C—Separating and combining—from 1 to 20.*

(i) Equal groups of objects.

(ii) Unequal groups of objects.

* See outline of plan of teaching, Nos. 1 to 20, p. 62.

D—Addition.

(i) Numbers from 1 to 20—Addition Table.

(ii) Numbers from 20 to 100—Types of Exercises.

(*a*)		6	7	7	7	
	6	6	6	7	7	
	6	6	6	6	7	
	—,	—,	—,	—,	—,	etc.

(*b*)	10	20	30	30	40	
	10	20	30	20	30	
	10	20	30	10	20	
	—,	—,	—,	—,	—,	etc.

(*c*)	20	30	60	90	
	8	8	8	8	
	—,	—,	—,	—,	etc.

(*d*)	10	20	40	70	
	18	18	18	18	
	—,	—,	—,	—,	etc.

(*e*)	25	35	55	75	
	8	8	8	8	
	—,	—,	—,	—,	etc.

(*f*)	25	35	55	75	
	18	18	18	18	
	—,	—,	—,	—,	etc.

(*g*)	10	20	40	60	
	15	15	15	15	
	18	18	18	18	
	—,	—,	—,	—,	etc.

(*h*)			10	15	
	9	6	12	9	
	8	7	13	17	
	7	2	25	16	
	6	9	19	6	
	5	8	7	13	
	—,	—,	—,	—,	etc.

While facility in adding may be regarded as a highly important end in teaching addition, it is not the only thing to be aimed at. The pupil must be trained to apply his knowledge, he must comprehend, and become familiar with, the common system of notation as a means of thought expression, and he must gain such mental power as will enable him to pass on to the next higher phase of relating quantities.

The question to be considered is, How are these results to be attained? Although this question may be viewed from several standpoints, we are here concerned with only one aspect of it, viz., the character of the exercises which the pupil is required to perform. They may be indicated as follows:

(i) Exercises requiring the pupil to use what he knows in discovering new facts. In (a), as given above, the pupil is required to relate the known, $6 + 6$, and $6 + 6 + 6$, to $6 + 6 + 7$, $6 + 7 + 7$, and $7 + 7 + 7$. After the pupil has performed the work, the teacher will question him in such a manner as to make him fully conscious of what he has done.

(ii) Exercises specially intended to familiarize the pupil with the ten-unit. Every primary teacher is aware of the rapid progress that a pupil is capable of making after he is once able to relate by tens. Why? Certainly not because the ten-unit possesses any magic power in itself. The reason is simply this: the decimal system of notation is an arbitrary one, which cannot, therefore, be employed by the pupil as a natural means of expression until he has become familiar with the unit which forms its basis. But just so soon as the pupil can think *in tens*, the resulting unification of thought and expression manifests itself, not only in the increased power which it gives, but also in the pleasure derived from the consciousness of such power. Much practice should therefore be given in finding the sum of such addends as will direct attention to the decimal system of notation. See exercises given above.

(iii) Exercises which pave the way for multiplication. Finding the sum of equal addends forms the most direct preparation for this.

(iv) Exercises which are designed to emphasize particular combinations. This is referred to elsewhere.

(v) General review exercises.

Nothing should be said about "carrying" until the pupil has actually practiced it. If the exercises are properly graded and related to one another, the pupil will almost unconsciously do the work for himself.

At the proper time, however, he should be made fully aware of how the operations he has already performed are related to the notation.

Before the formal process is taught the pupil should have a good knowledge of notation.

E—Subtraction.

(i) Numbers from 1 to 20.

(ii) Numbers from 20 to 100.

F—Multiplication.

(i) Numbers from 1 to 20.

(ii) Numbers from 20 to 100—Types of Exercises.

(*a*) The multiplicand a multiple of 10, as 10×2, 20×3, 20×4, 30×2, 30×3, etc.

(*b*) The multiplier a multiple of 10, as 2×10, 2×20, 3×10, 3×20, 3×30, 4×10, 4×20, etc.

(*c*) 12×2, 22×3, 22×4, etc.

(*d*) 15×2, 15×3, 25×4, etc.

(*e*) 2×15, 3×15, 4×25, etc.

(*f*) 24×2, 24×4, etc.

The exercises should be related in such a way that the pupil will discover the law of *commutation*. Objects should be used for this purpose: 3 blocks $\times$ 5 = 5 blocks $\times$ 3.

G—Division.

(i) Numbers from 1 to 20.

(ii) Numbers from 20 to 100.

Multiplication and division should be taught as related to each other. For example, 10 blocks $\times$ 3 = 30 blocks may be read as 10 blocks taken

3 times = 30 blocks, and it may be further interpreted as implying that 30 blocks contains 10 blocks 3 times. Multiplication and division are complementary to each other, and they should from the beginning be taught together. Even in giving practice in the more advanced work of a later stage it is well not to lose sight of this completely. When a pupil has found that $1728 \div 12 = 144$, he will understand it all the better for proving his answer.

H—Multiplication Table—from 1 to 100.

The pupil is to make his own table as he proceeds. Each number should be traced back to addition, at least for a time; thus, $6 + 6 + 6 + 6 + 6 = 30 = 5$ sixes.

MEASURING AND COMPARING.

A—Volume.

(i) Comparative size of objects—Estimating and testing by means of displacement of water when possible.*

(ii) Comparative size of rectangular piles of cubical blocks, leading up to definite measurement. The blocks should be of uniform size.

(iii) Relation of size to weight—Estimating and testing.

B—Weight.

(i) Comparative weight of objects—Estimating and testing by means of balance when possible.

(ii) Relation of weight to size—Estimating and testing.

C—Surface.

(i) Comparative area of plane surfaces of objects, as tables, desks, cubes, sheets of paper, pieces of cardboard, etc—Estimating and testing by superposition when possible.

*We are not certain as to the practicalness of this. It is merely suggested.

(ii) Comparative area of coloured circles, squares drawn on blackboard. (The whole figure must be chalked over, not merely the outline. Nothing should be said about linear dimensions here.) Estimating and testing by superimposing pieces of cardboard.

D—Length.

(i) Comparative length, breadth, height, etc., of objects—Estimating and testing.

(ii) Comparison of length with breadth, etc.—Estimating and testing.

(iii) Distance between objects—Estimating and testing by pacing.

E—Value.

(i) Measurement of sums of money as 10c., 15c., 20c., etc., using one-cent coins (afterwards five-cent coins, ten-cent coins, etc.).

(ii) Comparative value of small coins.

(iii) "Making change," etc.

(iv) Easy problems.

F—Time.

(i) Comparative length of intervals between successive acts, as ringing the bell, raising the hand, etc.

(ii) Comparative length of time taken by different objects in falling, as hail and snowflakes, pebbles and feathers, etc.

The main purposes of these lessons are (*a*) to give the pupil a clear conception of what is meant by volume, surface, etc., so that he may be

prepared to measure them; (*b*) to direct attention to the idea of quantitative measurement; (*c*) to train the pupil to observe closely, to measure with the eye, etc.

EXPRESSION.

A—Oral.

(i) Names of numbers from 1 to 100.

(ii) Language of number used in explaining or illustrating operations performed.

B.—Written.

(i) Figures.

(ii) Notation and numeration—Numbers from 1 to 100.

(iii) Symbols of operation.

(iv) Roman notation from 1 to 20.

(v) Exercises.

PROBLEMS.

A—Addition.

(i) One-step—John has 4 marbles and James has 5; how many have both together.

(ii) One-step—John has 4 marbles and James has 5 more than John; how many has James?

(iii) Two-step—John has 4 marbles and James has 5 more than John; how many have both together?

B—Subtraction.

(i) One-step—John had 6 marbles and he lost 2; how many had he left?

(ii) One-step—John has 8 marbles and James has 15; how many more has James than John?

(iii) Two-step—John won 12 marbles and lost 6. He then had 18; how many had he at first.

The problems given at this stage should be chiefly based on addition and subtraction. Until the pupil has grasped the idea of ratio he has but little power in the application of the processes of multiplication and division to the solution of problems.

Drill Exercises—Generally without objects.

A—Adding to or subtracting from a given initial number by twos, threes, fours, etc. This forms an excellent exercise on the different combinations.

B—Adding rapidly such as $\begin{array}{r}3\\ \underline{4}\end{array}$ $\begin{array}{r}9\\ \underline{8}\end{array}$ $\begin{array}{r}17\\ \underline{25}\end{array}$ $\begin{array}{r}36\\ \underline{44}\end{array}$ $\begin{array}{r}81\\ \underline{19}\end{array}$

C—Adding columns. The columns should be arranged, generally speaking, so that difficult combinations would be repeated frequently.

D—Finding the value of a series of numbers connected by the signs of operation, as, $64 - 32 + 17 - 8 + 3 - 12 =$?

E—Adding columns purposely designed to give practice in all possible combinations within certain limits. For example, this will furnish 16 different exercises in addition, presenting nearly all the combinations of the first 6 numbers.

6	5	4	2
5	2	3	6
3	4	2	3
4	3	6	5

F—Performing operations indicated as follows:

(i) $\left.\begin{array}{r}3\\5\\8\\9\\6\\4\end{array}\right\} + 14$ (ii) $\left.\begin{array}{r}18\\12\\15\\9\\6\\3\end{array}\right\} \times 3$ (iii) $\left.\begin{array}{r}20\\12\\16\\28\\24\end{array}\right\} \div 4$ (iv) $24 = \left\{\begin{array}{r}6\times\\3\times\\8\times\\4\times\\12\times\\2\times\end{array}\right.$ (v) $4 = \left\{\begin{array}{r}24\div\\48\div\\40\div\\16\div\\28\div\\32\div\\12\div\end{array}\right.$

Such exercises as the foregoing may be varied to any extent to suit the requirements of the pupil. They should be performed accurately, and as rapidly as possible. In so far as the pupil is able, he should be required to relate results. Why is 24 equal to *four* times 6, but *eight* times 3? Why are 24 divided by *six* and 48 divided by *twelve* equal to each other?

It is an excellent plan for the teacher to prepare a set of number charts for the purpose of drill. Exercises carefully thought out are likely to be much more effective than those hurriedly placed on the blackboard. The charts may be made of ordinary manilla paper, and if properly mounted they will last for a long time.

Outline of Work for Second Period.

Numbering—1 to 1000.

A—Addition, Subtraction, Multiplication and Division.

Plenty of practice should be given in addition and subtraction; but multiplication and division should become more and more prominent as the pupil advances.

B—Multiplication Table—From 1 to 144.

Measuring and Comparing.

A—Volume.

(i) Measurement of quantities of sand, water, etc., using pint, quart, and gallon measures—one measure at a time. Comparison of whole quantity with the unit. Comparisons of quantities measured. Problems.

(ii) Measurement of one standard by means of another, as finding the number of pints in a gallon, quarts in a gallon, etc. Comparisons based on such as 2 gallons = ? quarts, 2 gallons = ? pints. Relation of pint and quart without actual measurement, etc. Other standards. Problems.

(iii) Measurement of rectangular solids.

Rectangular blocks of uniform size should be first used. These can be arranged together so as to represent rectangular solids. The aggregates may be measured, taking one block, two blocks, etc., as the unit, as the pupil may choose.* Comparisons will then be made. The great difficulty which is usually experienced with cubic measure arises from the fact that standard units of solidity are derived from corresponding units of length. For that reason nothing should be said about linear measurement, at least in this connection, until the pupil has grasped the idea of solidity, and until he is able to select units which are applicable to the measurement of solids.

B—Weight.

(i) Weighing quantities, using ounce and pound weights. Comparison. Problems.

(ii) Comparison of ounce, pound, hundred weight. Problems.

(iii) Relation of weight to volume. A pint measure of water weighs so much, what will a gallon weigh? etc. Problems.

C—Surface.

(i) Measurement of rectangular plane surfaces, using one square foot as the unit. Comparison. Problems.

(ii) Measurement of a square yard of surface by means of the unit one square foot. Also the measurement of a square foot by the units, 1 square inch, 2 square inches, 36 square inches, 72 square inches, etc. Comparison. Problems.

(iii) Relation of different standards to one another. Problems.

A piece of cardboard exactly 1 foot square may be cut out neatly to represent the unit. By this means the attention of the pupil will be fixed on the idea of surface measurement. The main thing is to lead the pupil to see that a surface can be measured only by some unit of surface. The

*The teacher will observe that the operation first performed is one of analysis. Why?

form of the unit should be changed; for instance, the square may be cut into two equal parts, and the parts placed so as to form an oblong 4 times as long as wide, etc. If one unit is thoroughly taught, and if the pupil once sees that it is applicable to the measurement of surface, the main difficulty of square measure is overcome. What has been said regarding cubic measure applies equally to square measure.

D—Length.

(i) Measurement of linear dimensions of objects, as table, desk, floor, wall, blackboard, etc. Comparison. Problems.

(ii) Measurement of distances between objects, as two pickets placed in the ground 30 feet apart. Comparison. Problems.

(iii) Relations of the standards, inch, foot, yard, rod, to one another. Other standards. Problems.

E—Value.

(i) Measurement of amounts of money by means of standards, as cent, 5-cent piece, 10-cent piece, 25-cent piece, etc. Comparison. Problems.

(ii) Comparison of values of different coins. Problems.

(iii) "Making change." Problems.

F—Time.

(i) Measurement of time taken for the pacing of distances, the oscillations of a pendulum, etc. Comparison. Problems.

A weight suspended by a thread will be found very serviceable for this work, as the time of each oscillation may be varied to suit the circumstances, by shortening or lengthening the distance from the point of suspension to the centre of gravity of the weight. A clock or watch will be used to indicate the standard units of measurement, as second, minute, etc.

(ii) Relation of second, minute, hour, day, etc. Comparison. Problems.

Lessons on the sun dial and face of the clock will be found profitable.

FACTORS.

A—Factors of numbers—From 1 to 144.

B—Common factors of two or more numbers.

MULTIPLES.

A—Easy multiples of numbers.

B—Common multiples of two or more numbers.

FRACTIONS.

A—Exercises on division into equal parts.

B—Comparison of the whole and one of the parts, taking the part as the standard.

C—Comparison of the whole and one of the parts, taking the whole as the standard.

D—Easy fractional relations, as halves, fourths, eighths, thirds, sixths, ninths.

This work should not be begun until the pupil is able to form a proper conception of the word *times;* that is, as implying the ratio of one quantity to another. It is to be noted that the pupil will use this term long before it conveys to him its full meaning. He may say 3 times 6 feet = 18 feet, just as he would say 3 times 6 apples = 18 apples, almost at the commencement of the course. There is a long step between 6 feet × 3 = 18 feet, as meaning 6 feet + 6 feet + 6 feet = 18 feet, and 6 feet × 3 = 18 feet, as implying that 18 feet is a quantity 3 times as great as 6 feet. In the former case the unit may be regarded merely as one of the like things which make up the whole, while in the latter it must be regarded as a *measured quantity.* The teacher should satisfy himself that the pupil has actually made a comparison of the quantitative values of 6 feet and 18 feet. When the pupil has done this, however, he has the thought in mind which finds one form of its expression by means of a fractional number.

Expression.

A—Notation and Numeration—100 to 1000.

B—Roman Notation—20 to 100.

Problems—Types.

A—Counting by fives and tens. (Introductory.)

(i) I bought 20 yards of cloth at one store and 30 at another. How many yards did I buy? How much more is there in one piece than in the other? If I sell both pieces at $2 per yard, how much shall I get for them? I divide the money equally among 20 men, how much does each get? If there had been twice as many men how much would each have got?

(ii) John sold 20 quarts of milk in the forenoon and 30 pints in the afternoon. How many pints did he sell altogether? How much more did he sell in the forenoon than in the afternoon? At 5c. a quart, how much money did he get? If John's money is made up of 5-cent pieces, how many has he? How much cloth can he buy with it at 25c. per yard?

B—Counting by sixes and twelves. (Introductory.)

(i) In 12 yards how many feet?

(ii) In 48 feet how many yards?

(iii) Measure the length of the table, using a foot rule. What is its length in inches?

(iv) In 48 yards how many 6 feet measures?

(v) What is the value of 18 lbs. of rice at 6c.? How much butter at 12c. could be bought for the same money?

C—General Exercises.

(i) How many quarts in 3 gallons? In 3 gallons and 3 quarts? Compare 3 gallons and 3 quarts. At 5c. per quart find the value of 3 quarts. Find the value of 3 gallons. Compare 15c. and 60c. with 5c. How many times is 15c. contained in 60c.? How much greater is 60c. than 15c.? How many times greater? At 15c. per hour, how long would it take to earn 60c.?

(ii) How many yards in 18 feet? In $19\frac{1}{2}$ feet? In 72 inches?

(iii) I had 15c. and I got 45c. more. How many pencils at 5c. each did I buy with one-half of my money? I spent the other half in buying oranges at 3 for 10c., how many did I get? I sell the oranges at 5c. each. How much do I get for them? How much must I add to what I have in order to make it $1? How many oranges at 3 for 10c. can I buy for $1? How many at 3 for 5c.? Why are there twice as many oranges in the latter case?

In giving problems, the following points should be noted:

1. The problems should usually be related to the exercise preceding them.

2. They should be related to one another. It is generally profitable to give a series of problems bearing on one another and developing one main idea.

3. They should call forth the pupil's best effort. Problems or other exercises which are too easy are useless. Of course the opposite error must be guarded against also.

4. They should be carefully graded according to degree of difficulty.

In the foregoing, the primary object is to give the pupil a clear conception of a *measured* unit. This is done in two ways, (*a*) by presenting the unit so that attention may be directed to it as a measured quantity, (*b*) by requiring the pupil to measure suitable aggregates by means of it. All measurements should be performed as accurately as possible. It is not the *amount* of work done that counts most for either the purposes of number or the formation of right habits, it is the *character* of the work. Whenever a measurement is made, it should become the basis of a series of questions, leading the pupil to make comparisons in so far as he is able to do so.

The pupil should be allowed to use one unit until he has a good working knowledge of it, before another is introduced. New units should be related to those already known. Suppose, for example, that the pupil has become familiar with 1 foot as a unit of measurement by using it in determining such lengths as 12 feet, 16 feet, etc. Let him measure off say 12 feet, place in his hand a yard stick, saying nothing about its length, and ask him to measure the distance of 12 feet with it. He finds that the yard measure is contained 4 times in 12 feet. Question him so as to lead him to compare the length of the new unit with that of the old one. This is not by any means the easiest way to get the pupil to see the relation between the units, but it may be the best way, provided he has gained sufficient power to make the necessary comparison. But if he has not the power, it would be a serious mistake to try to force him to a conclusion. The stimulus should be as great as the mental condition of the pupil will warrant, but no greater.

Outline of Work for Third Period.

Fundamental Rules.

A—Addition, Subtraction, Multiplication and Division. Practice to secure accuracy and rapidity.

B—Multiplication Table. From 1 to 400.

C—Factors and Multiples—oral exercises.

Weights and Measures.

A—Tables as required for exercises.

B—Transformation of denominate units.

(i) Simple—one denomination.

(*a*) Reduction descending
(*b*) Reduction ascending
} Relate to each other.

(*c*) Problems—oral and written.

(ii) Compound—two or more denominations.

(*a*) Reduction descending
(*b*) Reduction ascending
} Relate to each other.

(*c*) Problems—oral and written.

Weights and measures not in common use should be omitted.

C—Compound Rules.

(i) Addition
(ii) Subtraction
} Relate to each other.

(iii) Multiplication
(iv) Division
(*a*) Finding the value of each part
(*b*) Finding the number of times
} Relate to each other.

(v) Problems—oral and written.

D—Commercial transactions.

(i) Evaluation. Finding values in connection with everyday business transactions.

(ii) Bills and accounts. Making out and receipting bills of goods as groceries, dry goods, hardware, farm produce, etc., in proper business form.

In all the foregoing exercises care will be taken not to introduce too many large numbers, as they make the work burdensome and serve no educational or practical purpose. In teaching principles small numbers should always be employed.

E—Measurements.

(i) Length.

(*a*) Practical measuring of distances — finding the length of lines, the dimensions of rectangular surfaces and solids.

(*b*) Comparison of lengths of lines, etc.

(*c*) Problems—oral and written.

(ii) Surface.

(*a*) Area of rectangles—land measurement.

(*b*) Comparison of areas of rectangular surfaces.

(*c*) Problems—oral and written.

(iii) Volume.

(*a*) Measurement of rectangular solids.

(*b*) Comparison of volumes of different solids.

(*c*) Problems—oral and written.

(iv) Practical applications.

(*a*) Fencing, ditching, tree-planting, etc.

(*b*) Carpeting, papering, painting, plastering, etc.

(*c*) Measurement of land.

(*d*) Measurement of lumber.

(*e*) Measurement of cordwood, stone, stonework, brickwork.

(*f*) Capacity of rectangular tanks, bins, etc.

(*g*) Problems.

In all of this practical work the pupil should find his own data by actual measurement as far as possible.

F—General review exercises and problems.

MEASURES.

A—Integral factors as *measures* of quantities.

(i) Denominate units (simple). Exercises.

(ii) Abstract units.* Exercises.

B—Common measures of two or more quantities.

(i) Denominate units (simple). Exercises.

(ii) Abstract units. Exercises.

C—Prime factors.

D—Greatest common measure of two or more quantities.

(i) By means of resolving numbers into prime factors.

(*a*) Exercises.

(*b*) Easy problems.

(ii) By means of the formal process.

(*a*) Exercises.

(*b*) Easy problems.

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* See page 7.

E—General review exercises and problems.

MULTIPLES.

A—Quantities which can be measured exactly by other quantities.

- (i) Denominate units (simple). Exercises on divisors.
- (ii) Abstract units. Exercises on divisors.

B—Common multiples of two or more quantities.

- (i) Denominate units (simple). Exercises.
- (ii) Abstract units. Exercises.

C—Prime factors. Review.

D—Least Common Multiple of two or more quantities.

- (i) By means of resolving numbers into prime factors. Exercises.
- (ii) By means of formal process. See Hamblin Smith's Arithmetic, p. 48. Exercises. Easy problems.

E—General review exercises and problems.

FRACTIONS—COMMON.

A—Exercises on the comparison of quantities. (Review.)

B—Division of units into equal parts. (Review.)

C—Preliminary exercises—no formal process.

- (i) Comparison of the fractional unit with the prime unit, as ———————— is 4 times —— also —— is $\frac{1}{4}$ of ———————— ; 12 inches = 4 times 3 inches, also 3 inches = $\frac{1}{4}$ of 12 inches; 5 feet = 5 times 1 foot, also 1 foot = $\frac{1}{5}$ of 5 feet, etc.

Such complementary thoughts must be kept before the pupil in order that the true idea of a fraction may be developed. It is one thing to learn to manipulate fractions, but a quite different thing to think of the relations which fractions express.

(ii) Comparison of different fractional units with a common prime unit and then with one another, as 6 inches $= \frac{1}{2}$ of 12 inches $= 3$ inches $\times 2 = \frac{2}{4}$ of 12 inches $= 2$ inches $\times 3 = \frac{3}{6}$ of 12 inches. Therefore, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$. Exercises.

(iii) Comparison of multiples of the fractional unit with the prime unit, as 12 inches = twice 3 inches $\times 2$, also twice 3 inches $= \frac{1}{2}$ of 12 inches.* Therefore, twice $\frac{1}{4} = \frac{1}{2}$. Exercises.

(iv) Comparison of equal parts of the fractional unit with the prime unit, as 12 inches $= 12$ times $\frac{1}{3}$ of 3 inches, also $\frac{1}{3}$ of 3 inches $= \frac{1}{12}$ of 12 inches. Therefore, $\frac{1}{3}$ of $\frac{1}{4} = \frac{1}{12}$. Exercises.

The pupil should be given many exercises such as those indicated above before the formal teaching of the subject begins.

D—Formal study.

(i) Notation.

(ii) Changing from one form to another.

(*a*) Mixed numbers to improper fractions.

(*b*) Improper fractions to mixed numbers.

(iii) Changing from one denomination to another.

(*a*) Equivalent fractions having different denominators.

(*b*) Equivalent fractions in lowest terms.

(*c*) Equivalent fractions having a common denominator. Comparison.

*It is here assumed, of course, that 3 inches $= \frac{1}{4}$ of 12 inches is well known. This comes under (i).

(iv) Addition, subtraction, multiplication and division.

(v) Complex fractions. Easy exercises.

(vi) Practical problems.

E—General review exercises and problems.

FRACTIONS—DECIMAL.

A—Notation. Relation to the Arabic (decimal) system. Reading decimals. Exercises.

B—Changing from decimal to vulgar fractions and *vice versa.* Exercises.

C—Addition, subtraction, multiplication and division of decimals.

D—Practical exercises. Problems.

E—General review exercises and problems.

FRACTIONS—PERCENTAGE.

A—A particular case of fractions.

(i) Meaning of the term. Illustrations.

(ii) Relation of percentage to decimal and common fractions.

(iii) Exercises and problems.

B—General application—Exercises and Problems.

C—Particular applications.

(i) Trade discount.

(ii) Insurance, taxes, etc.

(iii) Commission.

(iv) Profit and loss.

(v) Simple interest.
(vi) Bank discount.
(vii) Partial payments.
(viii) Compound interest.
(ix) Stocks.
(x) Exchange.

D—General review exercises and problems.

PROPORTION.

(i) Simple, Exercises and problems.
(ii) Compound. Exercises and Problems.

INVOLUTION.

EVOLUTION.

(i) By inspection.
(ii) Formal process.
(iii) Application—Easy exercises.

GENERAL REVIEW.

16. Plan of Teaching Nos. 1 to 20.*

Two.

(*a*) $2=1+1$; $2-1-1=0$; $1\times 2=2$; $2\div 1=2$.

Three.

(*b*) $3=1+1+1$; $3-1-1-1=0$; $3=1\times 3$; $3\div 1=3$.

Four.

(*c*) (i) Counting—forwards and backwards; $4=1\times 4$; $4\div 1=4$.
(ii) Forming groups of two.
(iii) $4=2+2$; $4-2-2=0$; $4=2\times 2$; $4\div 2=2$.

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* In order to save space such combinations as $3+4$, $5+2$, $7-2$, $11-5$, etc., are here omitted. These are to be fully dealt with as explained elsewhere.

It will be understood, of course, that concrete units are to be dealt with at the beginning. See page 18.

Five.

(*d*) (i) Counting—forwards and backwards.

(ii) $4+1$; $2+2+1$.

Six.

(*e*) (i) Counting—forwards and backwards ; $6=1\times6$; $6\div1=6$.

(ii) Forming groups of three.

(iii) $6=3+3$; $6-3-3=0$; $6=3\times2$; $6\div3=2$.

(iv) Review (*c*) iii.

(v) $6=2+2+2$; $6-2-2-2=0$; $6=2\times3$; $6\div2=3$.

Seven.

(*f*) (i) Counting—forwards and backwards.

(ii) $6+1$; $3+3+1$.

Eight.

(*g*) (i) Counting—forwards and backwards.

(ii) Forming groups of four.

(iii) $8=4+4$; $8-4-4=0$; $8=4\times2$; $8\div4=2$.

(iv) Review (*c*) iii and (*e*) v.

(v) $8=2+2+2+2$; $8-2-2-2-2=0$; $8=2\times4$; $8\div2=4$.

Nine.

(*h*) (i) Counting—forwards and backwards.

(ii) Review (*b*) and (*e*) ii.

(iii) $9=3+3+3$; $9-3-3-3=0$; $9=3\times3$; $9\div3=3$.

Ten.

(*i*) (i) Counting—forwards and backwards.

(ii) Forming groups of five.

(iii) $10=5+5$; $10-5-5=0$; $10=5\times2$; $10\div5=2$.

(iv) Review (*c*) iii, (*e*) v and (*g*) v.

(v) $10=2+2+2+2+2$; $10-2-2-2-2-2=0$; $10=2\times5$; $10\div2=5$.

Eleven.

(*j*) (i) Counting—forwards and backwards.

(ii) $10+1$.

(iii) $5+5+1$.

Twelve.

(*k*) (i) Counting—forwards and backwards.

(ii) Review (*e*) i.

(iii) $12=6+6$; $12-6-6=0$; $12=6\times 2$; $12\div 6=2$.

(iv) Review (*c*) i and (*g*) ii.

(v) $12=4+4+4$; $12-4-4-4=0$; $12=4\times 3$; $12\div 4=3$.

(vi) Review (*b*), (*e*) iii and (*h*) iii.

(vii) $12=3+3+3+3$; $12-3-3-3-3=0$; $12=3\times 4$; $12\div 3=4$.

(viii) Review (*g*) v and (*i*) v.

(ix) $12=2+2+2+2+2+2$; $12-2-2-2-2-2-2=0$; $12=2\times 6$; $12\div 2=6$.

It is unnecessary to give a detailed analysis any further, as the general plan is easily seen. It may be briefly stated thus:

(i) The chief aim at the beginning is to give the pupil power *to number*. With that end in view the idea of determining the *how many* of an aggregate by means of a known unit is constantly kept before him. Beginning with counting (by ones) as the fundamental process in numbering, the pupil is led step by step to define aggregates by means of numbered units, thus proceeding from the indefinite to the definite, in accordance with the natural order of thought development. This accounts for the fact that composite numbers, which are easily measured by different units, receive more attention at first than prime numbers.

(ii) Numbers such as 7, 11, 13, etc., are not analyzed to any great extent at the commencement of the course, because to

deal with them in detail would only lead to the memorizing of facts before they are mastered—an error to be guarded against as carefully as that of committing algebraic or trigonometrical formulæ before they have been deduced from first principles. It must not be forgotten that numbering is essentially a form of mental activity.

But although prime numbers are not to be analyzed minutely, care must be taken not to overlook them entirely. Their position (ordinal) in the regular series must be brought clearly before the pupil. The exercises in counting objects, etc., suggested in the foregoing outline, if properly performed, cannot fail to secure this.

(iii) Such combinations as $7+5$, $9+7$, etc., are for the most part to be omitted at the very beginning for two reasons, in addition to those which may be inferred from the foregoing.

(*a*) They present difficulties which the pupil is not prepared to meet.

(*b*) They will serve for a most important purpose after the pupil has gained some conception of what numbering really is. He will be given the opportunity of relating $6+7$ say to the known combination $6+6$ or $7+7$. $5+7$ may be related to $5+5$, $6+6$, or $7+7$, all of which are well known. These exercises will not only afford pleasure by occasioning appropriate mental activity, but they will also greatly increase the pupil's power to analyze and compare quantities. No attempt to get the pupil to memorize these should be made at first. The remembering of facts may be allowed to take care of itself for a time at least, provided the right kind of *thinking* is done. There is little difficulty in committing to memory facts that are thoroughly understood. One of the weaknesses of primary number

teaching consists in drilling mechanically on facts which, in so far as the pupil is concerned, have never passed through the thought stage. The proper order is, first comprehending the idea, second, mastering the idea by making use of it frequently, third, drilling on it to secure perfect familiarity with it—the mechanical phase.* To reverse this order is to put the cart before the horse.

(iv) By the time 20 or 30 is reached the pupil will have acquired some grasp of the subject. He will then be in a position to make a fuller analysis of the lower numbers. This work should be carried on in no haphazard manner, but according to a well designed plan, leading the pupil to connect each new idea with others already in his possession. For example, such exercises as these taken in the order given are comparatively useless for the purposes of teaching: $6+7$, $5+4$, $14+8$, $9+11$, $23-8$, etc. The pupil cannot see any connection which one part has with another, therefore the exercise can have little value except as a test of what is known. How different is the effect when a lesson is arranged so as to develop one leading thought, as for example:

$6+7$, $17+6$, $23-7$, $16+7$, $33-6$, $37+6$, $16+17$, $23-17$, $33-16$, $43-17$, $26+27$, $43-26$, and $37+26$.

This exercise is valuable because by performing it

(*a*) The pupil has been led to relate numbers to one another as $6+7$, to $16+7$, etc., and he has consequently gained power.

(*b*) The combination $7+6$ means more to him than it did before, for he has made a wider application of it.

(*c*) He has made progress in addition and subtraction.

*Stress should be laid on the first and second of these rather than on the third. Drill exercises, although quite necessary, if carried beyond a proper limit have a deadening effect.

(*d*) He has done at least something towards forming the habit of relating one thing to another.

The numbers given above extend beyond the stated limit of 20 or 30 in order to show how the primary lessons may be connected with those more advanced.

The pupil should be encouraged to discover different units in numbering an aggregate. When 16 blocks, say, are presented for the first time it is best to simply ask how many there are. Although an order for the selection of units is laid down in the foregoing outline, it is not intended that such order should be forced upon the pupil. One-half of the value of the lesson will be lost if he is not allowed to determine units for himself. In fact very much of the difficulty of arithmetic consists in recognizing units as being applicable to the measurement of quantities.

(v) At the same time that the analysis of the lower numbers is being carried on there will be progress made in connection with the higher, proceeding as before, from outline to detail. The pupil will pass gradually from 20 to 100, counting by tens, then by fives. The relations established between numbers under 20 should be immediately applied to those above it. Whenever the pupil learns that $7+3=10$ he should be required to use this knowledge in so far as he can. He will have a better grasp of $7+3$ when he has related it thus, $17+3$, $27+3$, etc., than he could possibly have had before doing so. Every new idea must be *used* in order to be fully mastered.

The exercises in multiplication and division indicated above should be based on corresponding ones in addition and subtraction. The latter processes will therefore precede the former. It is not the intention here to advocate a return to the old method of teaching the fundamental rules, either independently of one another, or in consecutive order; but it is

the intention to advocate their presentation in accordance with the logical sequence of their development. The pupil should be able to add 3 pears, 3 pears and 3 pears readily before he is asked to perform even the additional operation of counting the three addends. If this is true as to the lower, what must be said with reference to the higher phase of multiplication? The same principle applies in the case of subtraction and division. The prevailing practice of teaching addition, subtraction, multiplication, division and fractions (?) simultaneously, from the very commencement of the course, is based on the fundamental error of assuming that the processes of thought which these involve are co-ordinate rather than correlative.

The course here outlined is, for the most part, suitable for the average pupil of 7 years of age. Any work in number done with pupils below this limit should be very elementary indeed, such as counting and forming small groups of objects, stick-laying, drawing, etc. No regular number work should be prescribed for pupils below the age referred to for two reasons:

(i) They have not, generally speaking, attained to the maturity of thought necessary to enable them to deal with the subject intelligently or profitably.

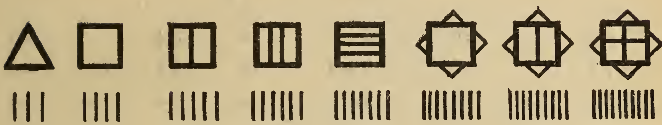
(ii) There is plenty of other school work which is well suited to them—reading, drawing, nature study, oral composition. Little children delight in these, because they can be adapted to their mental condition.

17. Seat Work for Junior Classes.

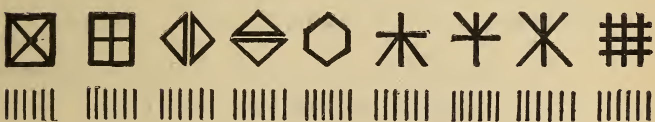
A—Objects.

(i) Arranging shoe-pegs, sticks, etc., into groups of two, three, four, five, etc.

(ii) Laying sticks or pegs so as to represent forms placed on blackboard, arranging below each form the number of objects used, thus:



(iii) Arranging sticks, pegs, etc., by threes, fours, fives, etc., so as to represent as many different forms as possible, thus:



(iv) Arranging sticks, pegs, etc., into groups of ones, twos, threes, fours, etc., numbering the groups in consecutive order, thus:

(v) Forming bundles of toothpicks, etc., by tens, and afterwards making up such numbers as 16, 26, 36, 46, etc., by adding, and also by taking away.

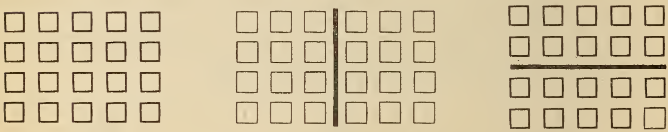
(vi) Illustrating by means of pegs, toothpicks, etc., operations indicated on blackboard, thus: (See No. iv.)

$1+0 = ?$	$2+0 = ?$	$3+0 = ?$
$1+1 = ?$	$2+2 = ?$	$3+3 = ?$
$1+1+1 = ?$	$2+2+2 = ?$	$3+3+3 = ?$
$1+1+1+1 = ?$	$2+2+2+2 = ?$	$3+3+3+3 = ?$
$1+1+1+1+1 = ?$	$2+2+2+2+2 = ?$	$3+3+3+3+3 = ?$

$1\times 1 = ?$	$2\times 1 = ?$	$3\times 1 = ?$
$1\times 2 = ?$	$2\times 2 = ?$	$3\times 2 = ?$
$1\times 3 = ?$	$2\times 3 = ?$	$3\times 3 = ?$
$1\times 4 = ?$	$2\times 4 = ?$	$3\times 4 = ?$
$1\times 5 = ?$	$2\times 5 = ?$	$3\times 5 = ?$

(vii) Illustrating by means of blocks operations already performed on slate, thus:

$5\times 4 = 4\times 5$ $\quad$ $6\times 4 = 3\times 8$ $\quad$ $5\times 4 = 10\times 2$



(viii) Laying one-inch, two-inch, three-inch, etc., sticks end to end, so as to represent lines of given length as 6 inches, 1 foot, 1 foot 6 inches, etc. Comparison.

(ix) Finding the measurement of sticks numbered 1, 2, 3, 4, etc., by means of one-inch, two-inch, etc., measures and writing results on slates. Thus stick No. 4 contains the two-inch measure 8 times, therefore it is 2 in. $\times 8 = 16$ inches long. Comparison.

(x) Determining the lengths of sticks by placing small measures end to end beside them. Thus stick No. 7 requires 6 four-inch measures to make up its whole length, it is therefore 4 in. $\times 6 = 24$ inches long. Questions as, how many 8 inches contained in the length of stick No. 7? How often must a three-inch measure be repeated to make up its length, etc. Comparison.

(xi) Laying sticks of measured length so as to enclose 2 sq. inches, 3 sq. inches, 6 sq. inches, 1 sq. foot, etc. Corresponding oral or written statements. Comparison of one unit, 2 units, 3 units, etc., with the whole area enclosed.

The unit of surface must be made clear to the pupil.

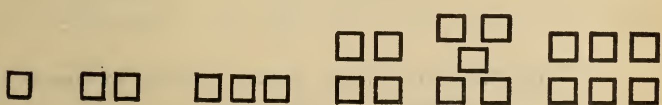
(xii) Laying sticks of measured length so as to enclose a stated area, giving it different forms.

(xiii) Forming cubes and other rectangular solids of given size by means of cubical blocks. Comparison.

In all of the preceding either the idea of counting or measuring should be prominent.

B—Representation of objects.

(i) Representing on slates, pegs, sticks, blocks, etc., by twos, threes, fours, etc., as:



(ii) Copying from blackboard drawings, illustrating operations, the pupil determining the right-hand side of the equation, as .

$$\Delta\Delta + \Delta\Delta = ? \qquad \mathrm{AAA} + \mathrm{AAA} = ?$$

$$\mathrm{TTTT} + \mathrm{TT} = ? \qquad \mathrm{HHHH} - \mathrm{HH} = ?$$

(iii) Using figures in addition to that given in No. (ii), as :

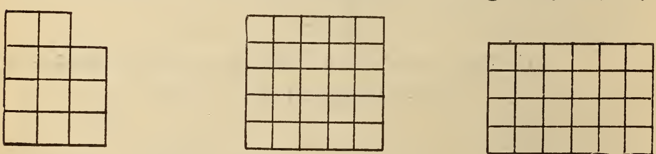
□ □ □ + □ □ □ = ?

3 + 3 =

(iv) Illustrating exercises, problems, etc. (first performed with figures), by means of circles, dots, squares, etc., as :

$$5 + 4 = 9 \qquad 4 + 4 + 1 = 9 \qquad 4 \times 3 = 3 \times ?$$

$$5 + 5 + 4 = \qquad 5 + 5 + 5 + 5 + 5 = \qquad 6 \times 4 = 4 \times ?$$



(v) Measuring lines drawn on slate by means of sticks, etc. Comparison.

(vi) Drawing on slate squares and rectangles of given area. Comparison of areas.

(vii) Representing rectangles of given area in different forms, etc. Comparison.

C—No objects.*

(i) Addition.

(*a*) Adding by ones, twos, threes, fours, etc., the results being expressed by figures, as:

2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, etc.

4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, etc.

10, 20, 30, 40, 50, 60, 70, 80, 90, 100, etc.

15 ft., 25 ft., 35 ft., 45 ft., 55 ft., 65 ft., 75 ft., 85 ft., etc.

13 pts., 33 pts., 53 pts., 73 pts., 93 pts., etc.

25 ds., 50 ds., 75 ds., 100 ds., etc.

(*b*) Adding numbers arranged in columns, as:

4	14	24	7	17	27	
4	4	4	4	4	4	
—,	—,	—,	—,	—,	—,	etc.
9	19	29	39	49	59	
8	8	8	8	8	8	
—,	—,	—,	—,	—,	—,	etc.
9	19	29	39	49	59	
9	9	9	9	9	9	
8	8	8	8	8	8	
—,	—,	—,	—,	—,	—,	etc.
9	19	19	27	39	49	
9	19	19	19	19	19	
8	8	18	18	18	18	
8	8	18	18	16	17	
—,	—,	—,	—,	—,	—,	etc.

* Standard units of measurement, as ft., yd., pt., wk., etc., should be kept prominently before the pupil.

(*c*) Adding numbers connected by signs:

2+2=?	4+4=?	8+8=? etc.
2+3=?	3+4=?	8+9=? etc.
6+4=?	16+4=?	26+4=?
10+9+1 =?	20+ 9+ 1 =?	30+ 9+ 1 =?
7+7+8 =?	7+ 7+ 8+ =?	7+ 7+ 7+ 7+ 8=?
7+8+9+10=?	10+11+12+13=?	14+15+16+17+18=?

(*d*) Problems based on preceding exercises.

(ii) Subtraction.

(*a*) Subtracting by ones, twos, threes, fours etc., from a given initial number, as:

12, 10, 8, 6, 4, 2.

30, 28, 26, 24, 22, 20, 18, 16, 14, 12, etc.

100, 96, 92, 88, 84, 80, 76, 72, 68, 64, etc.

150, 140, 130, 120, 110, 100, 90, 80, 70, 60, etc.

165 ft., 160 ft., 155 ft., 150 ft., 145 ft., 140 ft., etc.

(*b*) Finding the difference between numbers arranged in columns, as:

8	18	28	38	48	58
6	6	6	6	6	6
—	—	—	—	—	—, etc.
15	25	35	45	55	65
7	7	7	17	17	27
—	—	—	—	—	—, etc.
15	25	35	25	35	45
8	8	8	18	18	18
—	—	—	—	—	—, etc.
172	172	172	172	172	
4	14	24	34	44, etc.	
—	—	—	—	—	

(*c*) Performing operations indicated by signs, as:

9 – 6 = ? 19 – 6 = ? 29 – 6 = ? 39 – 6 = ? etc.

13 – 7 = ? 23 – 7 = ? 33 – 17 = ? 43 – 17 = ? etc.

75 = 15 + ? 75 = 25 + ? 75 = 45 + ? 75 = 65 + ?

	30 = ?		5 = ?
	34 = ?		15 = ?
	32 = ?		35 = ?
	20 = ?		65 = ?
40 –	24 = ?	125 –	75 = ?
	22 = ?		105 = ?
	10 = ?		115 = ?
	14 = ?		125 = ?
	12 = ?		

(*d*) Problems based on preceding exercises.

(iii) Addition and Subtraction.

(*a*) Adding to and subtracting from a given number, by twos, threes, fours, fives, etc., as:

50	55,	60,	65,	70,	75,	80,	85,	90,	95,	100.
	45,	40,	35,	30,	25,	20,	15,	10,	5,	0.
90	99,	108,	117,	126,	135,	144,	153,	162,	171,	180.
	81,	72,	63,	54,	45,	36,	27,	18,	9,	0.

When pupils are sufficiently advanced they should be required to account for differences, as, 99 – 81, 108 – 72, 117 – 63, etc.

(*b*) Performing operations indicated by signs, as:

6 + 6 – 3 = ? 6 – 3 + 6 = ? 6 + 3 = 6 – ? 6 + 6 = 3 – ? 9 + 12 – 5 – 4 = ?

9 + 12 = 5 + 4 + ? 9 – 5 = 12 + 4 – ? 25 + 30 – 15 + 10 = ?

25 – 15 + 10 + 30 = ? 25 – 15 = 30 – 10 – ? 25 – 15 = 30 – 10 – ?

These exercises are of little value as a means of securing facility in addition and subtraction. Their purposes are mainly, (1) to stimulate thought activity; (2) to familiarize the pupil with the signs of operation.

The former of these is too often lost sight of. A proper time-limit should be set for the performance of all such exercises.

(*c*) Problems involving addition and subtraction.

(iv) Multiplication.

(*a*) Making tables, as:

$2\times 1=2$	$3\times 1=3$	$4\times 1=4$	$5\times 1=5$	$6\times 1=6$
$2\times 2=4$	$3\times 2=6$	$4\times 2=8$	$5\times 2=10$	$6\times 2=12$
$2\times 3=6$	$3\times 3=9$	$4\times 3=12$	$5\times 3=15$	$6\times 3=18$
$2\times 4=8$	$3\times 4=12$	$4\times 4=16$	$5\times 4=20$	
$2\times 5=10$	$3\times 5=15$	$4\times 5=20$		
$2\times 6=12$	$3\times 6=18$			
$2\times 7=14$				
$2\times 8=16$				
$2\times 9=18$				
$2\times 10=20$				

(*b*) Multiplying numbers, as:

2	2	2	2	2	2	2	2	2
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
4	4	4	4	4	4	4	4	4
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
3	3	3	3	3	3	3	3	3
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
6	6	6	6	6	6	6	6	6
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
10	10	10	10	10	10	10	10	10
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—

16	16	16	16	16	16	16	16	16
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
20	20	20	20	20	20	20	20	20
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
26	26	26	26	26	26	26	26	26
2	3	4	5	6	7	8	9	10
—	—	—	—	—	—	—	—	—
6	6	6	6	6	6	6	6	6
20	30	40	50	60	70	80	90	100
—	—	—	—	—	—	—	—	—

In such exercises the true idea of multiplication should be kept in mind, viz., the repetition of the multiplicand so many times. For that reason one multiplicand (unit) should be kept before the pupil for a time. The exercises will be varied by changing the *multiplier*. The pupil should be required to compare products in so far as he is able, and account for their relation to one another. When the *multiplicand* is changed the new one should bear to the old a relation which the pupil can appreciate. For example, 2 might be followed by 4 or 6, 7 by 14 or 21, etc.

(*c*) Performing operations indicated by signs, as:

$4 \times 3 = ?$	$3 \times 4 = ?$	$4 \times 5 = ?$	$5 \times 4 = ?$
$8 \times 3 = ? \times 6$	$3 \times 8 = 6 \times ?$	$8 \times 4 = ? \times 16$	$4 \times 8 = 2 \times ?$

	3 = ?		3 = ?		1 = ?
	5 = ?		5 = ?		2 = ?
	6 = ?		6 = ?		3 = ?
4 ×	8 = ?	8 ×	8 = ?		4 = ?
	9 = ?		9 = ?	25 ×	5 = ?
	10 = ?		10 = ?		6 = ?
	11 = ?		11 = ?		7 = ?
					8 = ?
					9 = ?

(*d*) Problems based on preceding exercises.

(v) Division.

(*a*) Making tables as :

$60 \div 10 = 6$		$1 = 72$
$60 \div 6 = 10$		$2 = 36$
$60 \div 5 = 12$		$3 = 24$
$60 \div 4 = 15$		$4 = 18$
$60 \div 3 = 20$	$72 \div$	$6 = 12$
$60 \div 2 = 30$		$8 = 9$
$60 \div 1 = 60$		$9 = 8$
		$12 = 6$

(*b*) Dividing numbers, as :

2)12	2)16	2)20	2)24	2)32
4)12	4)16	4)20	4)24	4)32
2)48	2)72	2)96	2)144	2)288
4)48	4)72	4)96	4)144	4)288
6)48	6)72	6)96	6)144	6)288
8)48	8)72	8)96	8)144	8)288
2)108	3)108	4)108	6)108	12)108

(*c*) Problems based on preceding exercises.

(vi) Multiplication and Division.

(*a*) Exercises as follows :

12 ft. $\times 3 = 18$ ft. $\times$? 12 ft. $\times 4 = 4$ ft. $\times$? 12 ft. $\times 6 = 3$ ft. $\times$?

$12 \times 6 = 144 \div$? $12 \times 6 \div 3 = 12 \times$? $12 \div 6 \times 8 =$? $\times 2$.

$100 \times 2 \div 50 = 16 \times 2 \div$? $100 \div 2 \times 8 =$? $\times 2 \times 4$.

(*b*) Problems involving multiplication and division.

(vii) Review exercises, involving the four simple rules. Problems.

Seat work may be exceedingly profitable, or it may be just the opposite. This will depend on the circumstances under which it is carried on. The following are the principal conditions for successful work:

(i) The exercises should be properly graded.

(ii) They should be related to one another, so that what is taught to-day may be used to-morrow.

(iii) Each exercise should, as a rule, develop one main idea —a particular combination or relation.

(iv) The exercises should stimulate thought activity. Generally speaking, they do not at all compare with class exercises for securing facility in calculating; they should, therefore, be constructed so as to make the pupil *think*.

(v) A proper time-limit must be set for the performance of an exercise. No more than a reasonable time should ever be allowed, otherwise sluggish thinking and acting will be the result.

(vi) Accuracy and neatness must be insisted upon. On no account should these points be overlooked. Habits which will affect the whole life of the pupil are being formed.

(vii) The pupil's work must be carefully supervised. Experience has proved this to be the case. It is better to allow the pupil to enjoy a game on the play ground, than to have him doing a school exercise to which the teacher cannot give proper attention, either during the time of its performance, or shortly afterwards.

In the foregoing, no attempt has been made to exhaust the different forms of exercises that will be found useful in actual practice. An effort has been made, however, to present only such types as are regarded the most valuable. Much of the seat work done by primary pupils under the designation of

"number busy work," such as representing or copying drawings from the blackboard, into which the element of counting scarcely enters, is comparatively useless except as a means of keeping pupils employed. Now, while the ability to provide a variety of attractive exercises for young pupils is one of the essential characteristics of the successful primary teacher, we cannot consider such ability a mark of the highest degree of excellence unless it be accompanied by such professional knowledge as is necessary to determine the educational value of the exercises given. To furnish "busy work" for the sole purpose of preventing pupils from getting into mischief, no matter how necessary it may be, is a tacit confession of weakness. All school exercises should be more than merely interesting or attractive, they should be educative. True interest, let it be remembered, is that which grows out of the study of a subject for its own sake.

Cards containing written exercises will be found convenient —especially when there is lack of blackboard space. They may be prepared by the teacher after school hours, thus saving time during the day. Each card should emphasize one main thought. After the cards are prepared they should be numbered, indexed, and filed, so as to be easily referred to.

Problems should be related to other exercises. A few types are here presented. They may suggest one or two useful lines of work.

1. John has 8 apples and James has 18, how many have both?

2. 16 pencils is how many more than 4?

3. Mary bought 5 pencils, lost 2 and gave away 1, how many has she left?

4. John bought 10 lbs. of sugar, and William bought 5 more than John. How much did both buy?*

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* Fixed standards of measurement should be prominent in problem work.

5. What must I add to 20 cents to have 35 cents?

6. A boy earns 10 cents every day, how much will he earn during the week?

7. How many gallons in 5 times 12 gallons?

8. How many 3 inches in 12 inches?

9. 3 times $15 contains how many $5 bills?

10. How many dollars in 400 cents?

11. A man earns 75 cents every forenoon and 50 cents every afternoon, how much will he earn during the week?

12. How many 2 square inches in 6 square inches?

13. How many inches in 2 feet?

14. How many quarts in 8 pints?

15. How many pints in 1 quart 4 pints?

16. 3 times 2 pints make how many quarts?

17. How many 3 pints in 3 quarts 3 pints?

18. How much is 2 feet greater than 15 inches?

19. Mark off 1 foot 6 inches into 3-inch lengths.

20. How many 4-inch measures are there in twice 1 foot 6 inches?

18. Practical Suggestions.

Reading Numbers.

(i) Read 44, as 4 tens, and 4 ones; 444 as 4 hundreds, 4 tens, and 4 ones. Questions on relation of position to value. Exercises.

(ii) Read 459, as 4 hundreds, 5 tens, and 9 ones. Questions. Exercises.

(iii) Read 459 in other ways as 4 hundreds, 4 tens and 19 ones; 3 hundreds, 15 tens, and 9 ones, etc. Exercises.

Writing Numbers.

(i) Write 300, 30, and 3. Add together. Read the sum as 3 hundreds, 3 tens, and 3 ones. Explain. Questions. Exercises.

(ii) Write 5 hundreds and 16 ones. Read in different ways. Exercises.

Addition.

(i) Add by twos, threes, fours, etc., from 1, 2, 3, 4, 5, 6, 7, 8, 9.

(ii) Read sums rapidly.

6	8	6	6	7	7	7	9	9	9
2	2	6	5	7	6	5	8	9	10
—	—	—	—	—	—	—	—	—	—

8	8	8	8	8	8	8	8	8	8
3	4	6	5	2	7	9	10	12	24
—	—	—	—	—	—	—	—	—	—

(iii) Add (introducing carrying).

			55		55
		55	55		55
		55	55		66
		—	—	55	66
22	44	10	15	55	88
22	44	10	15	55	88
—	—	—	—	—	—
44	88	110	165	165	418

(iv) Two column addition. Add the numbers 15, 65, 47, 84, thus: 15, 75, 80, 120, 127, 207, 211. This is an excellent exercise for senior pupils.

(v) Testing results.

28 + 29 + 30 + 31	118	342	
32 + 33 + 34 + 35	134	986	
36 + 37 + 38 + 39	150	735	2063
40 + 41 + 42 + 43	166	429	
44 + 45 + 46 + 47	182	834	
48 + 49 + 50 + 51	198	927	2190
228 + 234 + 240 + 246	948	4253	4253

(vi) Prove by casting out the 9's. This is the method generally employed by accountants.

Much practice should be given in adding columns. Let the pupil add short columns over and over again until he can add them rapidly. As progress is made the length of the columns should be increased. Each exercise should, generally, emphasize some combination, as 7 + 6, 27 + 6, 47 + 6, etc.

The fundamental combinations of addition should be mastered. They are quite as important as the multiplication table. Charts may be used with advantage for rapid drill work.

Subtraction.

(i) Count backwards, commencing at 50, by twos, threes, fours, etc.

(ii) Read remainders only.

16	18	20	26	28	30	36	48	50
8	8	8	8	8	8	8	8	8

(iii) Find the difference by adding to the less between 1000 and 120, 320, 840, 847, 850, 920, 940, 947, 950.

(iv) Making change. Much practice should be given by the ordinary business method.

It may be applied thus :

$10 − $6.85 = $3.15. ($5+5=10$, $9+1=10$, $7+3=10$; or better, $85+15=100$, $7+3=10$.)

$6497-3988=2509$. ($8+9=17$, $9+0=9$, $9+5=14$, $4+2=6$.)

(v) Steps leading up to formal process.

$$\begin{array}{rrrrrrrrr}
8 & 18 & 18 & 28 & 12 & 22 & 22 & 32 & 52 \\
4 & 4 & 14 & 14 & 9 & 9 & 19 & 19 & 39 \\
\hline
\end{array}$$

(vi) Proofs of correctness of work. (*a*) Adding remainder and subtrahend. (*b*) Casting out the 9's.

It is not well to spend much time in trying to get the pupil to comprehend the principles underlying the formal process until after he has acquired a knowledge of the relations of denominate units. The *use* of the process must be thoroughly understood.

Subtraction should be taught in its relation to addition.

MULTIPLICATION.

(i) Exercises leading up to formal process.

$$\begin{array}{r}
600\times 8=4800 \\
60\times 8=480 \\
6\times 8=48 \\
\hline
5328
\end{array}
\qquad
\begin{array}{r}
666 \\
8 \\
\hline
48 \\
480 \\
4800 \\
\hline
5328
\end{array}
\qquad
\begin{array}{r}
666 \\
8 \\
\hline
4800 \\
480 \\
48 \\
\hline
5328
\end{array}
\qquad
\begin{array}{r}
666 \\
8 \\
\hline
5328
\end{array}$$

(ii) In the following, compare the multiplicand and the different products. Account for relations of products.

		120
		3
	3 = 90	360
	5 = 150	4
	6 = 180	1440
30 ×	10 = 300	5
	12 = 360	7200
	15 = 450	6
	18 = 540	43200

(iii) The parts of the multiplication table* up to 144 which will require special drill are 18, 21, 24, 27, 28, 32 36, 42, 48, 54, 56, 63, 72, 84, 96, 108, 132.

(iv) Read the products only. Drill.

25	250	252	2520	2522
12	12	12	12	12
——	——	——	——	——

Drill exercises should not only secure facility in performing operations but they should also give increased power to relate quantities. Charts will be found valuable.

(v) Practical Methods. Compare the following:

	755	755
	16	16
	4530	755
	755	
(*a*)		12080
	12080	
	4365	4365
	523	523
	13095	13095
	8730	10039
	21825	
(*b*)		2282895
	2282895	

* After the multiplication table up to 12 × 12 is learned, much practice will be given in finding products up to 20 × 20.

(c)

$$\begin{array}{r} \$2.25 \\ 448 \\ \hline 1800 \\ 900 \\ 900 \\ \hline \$1008.00 \end{array} \qquad\qquad \begin{array}{r} \$896 \\ \$112 \\ \hline \$1008 \end{array}$$

(d)

$$\begin{array}{r} \$11.28 \\ 75 \\ \hline 5640 \\ 7896 \\ \hline \$846.00 \end{array} \qquad\qquad \begin{array}{r} \$1128 \\ \$\ 282 \\ \hline \$846 \end{array}$$

Short methods should not be dealt with until the pupil has comprehended the ordinary processes fully. At the proper time, however, such a problem as this will furnish an excellent means of discipline. Find the cost of 1728 cords of wood at $\$1.62\frac{1}{2}$ per cord. (Short method.) The pupil must, of course, devise the method himself. The importance of the law of *commutation* will be readily seen in this connection.

Division.

(i) Exercises leading up to short division.

$$\begin{array}{l} 6 \div 2 = 3 \\ 60 \div 2 = 30 \\ 600 \div 2 = 300 \end{array} \qquad 3\overline{)66} \qquad 3\overline{)666} \qquad 3\overline{)678}$$

Find the greatest number of hundreds, tens, and ones, exactly divisible by 4 in 80, 84, 85, 92, 95, 440, 560, 564, 567, 648, 650, etc. Questions and exercises.

(ii) Exercises leading up to long division. (To be performed by inspection.)

$18 \div 15 = ?$	$22 \div 21 = ?$
$36 \div 15 = ?$	$25 \div 21 = ?$
$45 \div 15 = ?$	$42 \div 21 = ?$

$50 \div 15 = ?$	$48 \div 21 = ?$
$60 \div 15 = ?$	$50 \div 21 = ?$
$66 \div 15 = ?$	$84 \div 21 = ?$
$90 \div 15 = ?$	$90 \div 21 = ?$
$100 \div 15 = ?$	$100 \div 21 = ?$
$150 \div 15 = ?$	$210 \div 21 = ?$
$160 \div 15 = ?$	$220 \div 21 = ?$
	$425 \div 21 = ?$

Find the greatest number of tens exactly divisible by 25 in 250, 255, 422, 525, 760, 840, etc.

Practical methods.

```
    396 ) 65923 ( 166          396 ) 65923 ( 166
          396                         2632
          ----                        2563
          2632                         187
          2376
          ----
           2563
           2376
(a)        ----
            187
```

(*b*) Such exercises as these will be found useful:

(*a*) Divide 1342 by 90 (factors).

(*b*) " " 21 (factors).

(*c*) " " 125.

(*d*) " " 225.

(*e*) " " $33\frac{1}{3}$.

Such as the foregoing are suitable for pupils who have made considerable advancement in the subject.

(*c*) Divide 1434227 (1) by 999; (2) by 998.

(1)

$$\begin{array}{r|r} 1434 & 227 \\ 1 & 434 \\ & 1 \\ \hline 1435 & 662 \end{array}$$

(2)

$$\begin{array}{r|r} 1434 & 227 \\ 2 & 868 \\ 1 & 4 \\ \hline 1437 & 101 \end{array}$$

This method cannot be regarded as very practical. It is applicable only in the case of divisors a little less than a power of 10.

Exercises in factoring and finding multiples should accompany work in multiplication and division. These exercises will greatly increase the pupil's knowledge of number. They may take such forms as the following:

(*a*) Find by inspection three factors of 64, 72, 96, 108, 210, 450, etc.

(*b*) Find three multiples of 15, 20, 25, 75, etc.

(*c*) Divide $12 \times 3 \times 5$ by 15; divide $12 \times 32 \times 64$ by $3 \times 8 \times 16$; divide $49 \times 50 \times 39$ by $13 \times 15 \times 21$, etc.

If the exercises are well chosen the pupil will soon discover that the work may be shortened by cancellation.

In teaching the simple rules, care must be taken not to allow the exercises to degenerate into performing merely mechanical operations. For example, the pupil must understand when he multiplies $10 by 12 he has either combined 12 addends of $10 each or he has found an amount 12 times as great as $10. The latter conception is the higher, in that it involves a comparison of the unit $10 with the amount found. The aim should be to present the simple rules so that the pupil will gain a true notion of number as expressing the ratio between the unit and the whole quantity. Until this stage of thought is reached, multiplication and division cannot

mean anything more than short processes of addition and subtraction respectively—a meaning which carries with it but little power.

Fractions.*

The term fraction seems to convey different meanings. It is used

(i) To express a ratio between two quantities, as 5 feet is $\frac{5}{6}$ of 6 feet. Here a direct comparison is made between the quantities 5 feet and 6 feet, and $\frac{5}{6}$ expresses the relation between them, 6 feet being taken as the standard of comparison. It is this mode of thought which is employed in the solution of the problem: If 6 pencils cost 12c., what will 5 pencils cost? when the solution assumes either of the forms

6 pencils cost 12c.,
$\therefore$ 1 pencil costs $\frac{1}{6}$ of 12c.,
$\therefore$ 5 pencils cost 5 times $\frac{1}{6}$ of 12c.;

or, 6 pencils cost 12c.,
$\therefore$ 5 pencils cost $\frac{5}{6}$ of 12c.

We see this the moment we answer the question, why does 5 pencils cost $\frac{5}{6}$ of 12c.? Because 5 pencils is $\frac{5}{6}$ of 6 pencils.

The first form of solution differs from the second only in one respect—an intermediate step is taken in order that the comparison may be more easily made. It is less difficult to establish a comparison between 1 and 6 and then between 5 and 1, than to establish it directly between 5 and 6. The point to be noticed is that in both cases the fractions express *ratio.*

(ii) To denote an unperformed process of division and multiplication. According to this idea 5 feet $=\frac{5}{6}$ of 6 feet,

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* For a complete treatment of fraccions see *Psychology of Number*, by McLellan and Dewey—an admirable book just published.

means that if we divide 6 feet into six equal parts and take five of them, the result will be five feet. The mode of thought in this case corresponds to that of finding the value of $\frac{5}{6}$ of 12c. in the preceding problem, after the ratio has been determined and stated. To find the value of $\frac{5}{6}$ of 12c., we divide 12c. into 6 equal parts and take 5 of them. This view does not enable one to use a fraction in the solution of a problem, but merely as a process in evaluating the result.

It is clear that when the pupil can perform the operations of multiplication and division intelligently he has but little to learn in order to deal with fractions used in this sense. If asked to divide 12c. into 6 equal parts and take 5 of them he will immediately do so, provided that his attention has been carefully directed to definite units of measurement, this being the necessary condition for the development of the true idea of fraction. But, it must be remembered, the difficulty of a fraction does not so much consist in finding the equal parts—the initial step—as in relating the parts to the whole. The latter implies establishing a ratio between a unit, or a number of units, and a whole quantity taken as the standard of comparison.

(iii) To denote a kind of concrete unit.* Suppose one yard to be divided into three equal parts, then we may give the name foot to each part. 1 part would be designated 1 foot, 2 parts 2 feet, etc. In the same way, if we allow ourselves to lose sight of the relational idea, we may call each part by the name third. One part would be called 1-third, 2 parts, 2-thirds, etc. Similarly, if we divide any other quantity into three equal parts we may give each part the name third. What is supposed to be a fraction is not a fraction at all, but a kind of concrete unit which has no fixed value. Unfortunately, much valuable time is wasted in drilling

* See The Public School Journal, vol. xiv., No. 10.

small children unable to number 10 objects on fractions of which they cannot possibly have any proper idea.

It is exceedingly important for the teacher to distinguish from one another the mental processes corresponding to the different cases given above. It is easy to fall into the error of assuming that because a pupil can manipulate fractional forms, which may mean to him, in reality, very little, he is able to make use of fractions in the solution of problems, or in other cases where they represent not merely arithmetical operations, but definite quantity relations.

As there is much difference of opinion regarding the time at which the teaching of fractions should begin, it may be well to indicate the ground on which such difference rests. The real question at issue is, how does the idea of fraction grow up in the mind? Does it arise from merely *numbering* acts of attention occasioned by objects, sounds, events, ideas, etc. (there can scarcely be any doubt that the idea of number—the *how many*—may be developed by any means that will occasion successive acts of attention), or from numbering definite quantitative units? In other words, is the fundamental idea that of *time* only, or does it involve both *space* and *time?* If numbering units, which are not quantitatively defined, can give rise to the idea of fraction, why should it be considered important in teaching fractions to deal with units which are quantitatively equal to one another? On the other hand, if both number and definite quantity are included, why should fractions be introduced until some progress is made in numbering measured units?

It is sometimes urged by those who favour teaching fractions from the beginning of the course in number that young children use the language of fractions intelligently in ordinary conversation. For example, a child may say that 50 cents is half a dollar. Certainly, there is nothing to prevent a five

year old boy from remembering two names for one thing. He will also remember that a dime is one-tenth of a dollar or one-fifth of 50 cents if told frequently enough. He may learn further that one-fifth is equal to two-tenths! If this is taught by means of objects he may even "catch the trick" of illustrating it after a fashion. Surely it is perfectly plain that all of this may be done and much more without involving a true conception of a fractional unit as related to a prime unit.

The teaching of fractions usually presents considerable difficulty. The cause of this may generally be traced back to one or more of the following:

(i) Attempting to deal with fractions at too early a stage.

(ii) Regarding fractions as indicating merely processes of multiplication and division.

(iii) Losing sight of the prime unit from which the fractional unit derives its meaning. This is closely allied to the last.

(iv) Introducing symbols before their meaning is fully developed.

(v) Showing the pupil how to perform operations with fractions instead of allowing him to discover rules and methods for himself.

Preliminary Work.

In teaching fractions the first step is to develop a proper notion of what a fraction is. This may be done by dividing strips of paper, lines, etc., into equal parts, and by dealing with denominate units. Compare a line 1 foot long with another 2 feet long; a line 2 feet long with another 4 feet long. Compare 5c. and 25c., 10c. and 100c., etc. It makes no difference whether the pupil answers that 25c. is 5 times 5c. or that 5c. is $\frac{1}{5}$ of 25c., provided that a comparison of the quantities has actually been made. The idea of fraction is implied in both, although expressed only by the latter.

The order of steps indicated by the foregoing may be more fully stated thus: (i) Separating the whole into parts (roughly) and *numbering* the parts. (ii) Dividing the whole into *equal* parts and numbering the parts. (iii) Comparing one or more of the equal parts with the whole; also, the whole with one or more of the equal parts. When the third stage is reached whole numbers are found insufficient to define all of the ratios which may be established; hence the necessity for fractional numbers. It will be seen that the very first mental effort made in numbering a quantity involves the idea of fraction as existing, not actually, but *potentially*. As the mind develops it demands a greater degree of accuracy for the satisfaction of its wants. The unit of measurement is made definite, and the necessary condition for the realization of the idea which at first existed only in possibility is afforded. As soon as fractions are dealt with in relation to the whole subject of number, many of the difficulties which are now experienced in teaching them will vanish. Work in fractions will not be prematurely forced upon the pupil, but it will be deferred until such time as the requirements of mental growth necessitate the comparison of quantities on the basis of the well-defined unit.

The second step is to give such exercises as will develop in the mind of the pupil the principles underlying all fractional work.

These principles may be stated as follows:

(i) Multiplying or dividing the numerator of a fraction by a number multiplies or divides the fraction by the number.

(ii) Multiplying or dividing the denominator of a fraction by a number divides or multiplies the fraction by the number.

(iii) Multiplying or dividing both terms of a fraction by a number does not alter the value of the fraction.

The method of dealing with principle i. may be illustrated thus: Compare $\frac{1}{4}$ of a foot and $\frac{2}{4}$ of a foot; $\frac{2}{6}$ of \$6 and $\frac{4}{6}$ of \$6; $\frac{2}{7}$ of \$14 and $\frac{6}{7}$ of \$14; $\frac{1}{5}$ of a pint and $\frac{4}{5}$ of a pint, etc.

Compare $\frac{2}{8}$ and $\frac{4}{8}$; $\frac{2}{8}$ and $\frac{6}{8}$; $\frac{2}{8}$ and $\frac{8}{8}$, etc.*

There should be no haste in getting the pupil to make a formal statement of the principle. The aim should be to give

* At first the pupil should be required to supply concrete units where no unit is given.

him *power* to make use of fractions, not simply to teach him how to perform operations. The power gained will be proportionate to the amount of well-directed, independent thinking which the pupil does in connection with exercises in comparing fractional quantities.

Formal Study.

The formal study of the subject may begin by giving the pupil practice in changing the form of a fraction without altering its value. No new principle is involved in this. To change fourths to twelfths is quite as easy as to change yards to feet, provided the relation between one-fourth and one-twelfth is as well known as that between one yard and one foot. Here, again, we see the necessity for exercises in comparison.

The reduction of fractions may take several forms of which the following may be regarded as types :—

(i) Find the number of fourths of a dollar in $\$1\frac{3}{4}$.

(ii) Find the number of dollars in 10 fourths of a dollar.

(iii) Find the number of fourths of a dollar in $\$\frac{5}{10}$.

(iv) Express $\$\frac{25}{40}$ in its lowest terms.

(v) Express $\$\frac{3}{4}$ and $\$\frac{4}{5}$ as fractions having a common denominator.

The exercises should be performed chiefly by the method of inspection. Viewed from either the standpoint of discipline or practical utility, it is well to give ample practice in easy fractions before proceeding to those involving large numbers.

The next step is to teach the application of the fundamental rules to fractions.

In introducing addition and subtraction, exercises should be given as: Find the value of $\$\frac{1}{4} + \$\frac{3}{4}$, $\$\frac{1}{3} + \$\frac{1}{4}$, $\frac{1}{4}$ pint $+ \frac{3}{5}$ pint, $\$\frac{1}{5} - \$\frac{1}{10}$, $\frac{1}{3}$ yd. $- \frac{1}{4}$ yd., etc.

The pupil should be required to point out the resemblance which adding and subtracting fractional units bears to corresponding operations which he has already performed with other units. He should frequently illustrate his work by means of objects. The representation of lines on the blackboard will be found both convenient and useful.

Multiplication and division present to the pupil greater difficulty than addition and subtraction. This is chiefly owing to the fact that the former involve a two-fold ratio.

(i) The ratio expressed by the fractional unit.

(ii) The ratio implied in the operation to be performed.

It is true that $\$\frac{1}{4} \times 3$ may mean nothing more to the pupil than $\$\frac{1}{4} + \$\frac{1}{4} + \$\frac{1}{4} = \$\frac{3}{4}$, which expresses directly but one ratio, viz. : that which the fractional unit $\$\frac{1}{4}$ bears to the prime unit \$1. But with only the thought of a single ratio in mind $\$\frac{1}{4} \times \frac{1}{3}$ must be meaningless. As soon, however, as the pupil grasps the idea that in $\$\frac{1}{4} \times 3 = \$\frac{3}{4}$, 3 expresses the ratio between $\$\frac{1}{4}$ and $\$\frac{3}{4}$, he is in a position to comprehend the meaning of $\$\frac{1}{4} \times \frac{1}{3} = \$\frac{1}{12}$.

In teaching multiplication and division of fractions much practice should be given in comparing quantities. Compare $\$\frac{1}{4}$ and \$2, $\$\frac{1}{4}$ and $\$\frac{3}{4}$, $\frac{1}{3}$ gal. and 5 gals., $\frac{1}{8}$ lb. and $2\frac{1}{2}$ lbs, etc.

By what must we multiply $\$\frac{1}{4}$ to get \$2? How is this expressed? What part of 5 gals. is $\frac{1}{3}$ gal.? Express this on blackboard.

The steps in teaching multiplication of fractions may be indicated as follows:

(i) Multiply a fraction by a whole number, as: $\$\frac{3}{4} \times 2$, pt. $\times 4$, $\frac{3}{10} \times 2$, $\frac{13}{16} \times 5$, etc.

(ii) Multiply a whole number by a fraction, as: $\$6 \times \frac{1}{2}$, $\$6 \times \frac{2}{3}$, 8 gals. $\times \frac{1}{4}$, $10 \times \frac{3}{5}$, etc.

(iii) Multiply a fraction by a fraction, as: $\$\frac{1}{2} \times \frac{1}{2}$, $\$\frac{2}{4} \times \frac{1}{8}$, $\frac{3}{4} \times \frac{8}{10}$, $\frac{4}{5} \times \frac{3}{8}$, etc.

The steps in teaching division are as follows:

(i) Divide a whole number by a fraction, as: $\$2 \div \$\frac{1}{2}$, 5 gals. $\div \frac{2}{3}$ gal., $5 \div \frac{2}{3}$, etc.

(ii) Divide a fraction by a whole number, as: $\$\frac{1}{2} \div 8$, $\$\frac{4}{5} \div 16$, $\frac{5}{6} \div 25$, etc.

(iii) Divide a fraction by a fraction, as: $\$\frac{1}{2} \div \$\frac{1}{4}$, $\$\frac{5}{6} \div \frac{7}{8}$, $\frac{10}{11} \div \frac{5}{22}$, etc.

A full interpretation of such an expression as $\$2 \div \$\frac{1}{2} = 4$, generally presents difficulty to the beginner. What are the meanings which it conveys? (The pupil's answer to this will depend altogether on his power of interpretation.)

(*a*) $\$\frac{1}{2}$ may be taken from \$2 four times, or to express the same idea in other words, \$2 contains 4 half-dollars.

(*b*) \$2 is made up of 4 equal parts of $\$\frac{1}{2}$ each.

(*c*) \$2 is four times as great as $\$\frac{1}{2}$.

These represent different stages in relating. In (*a*) the parts (units) are *numbered*, in (*b*) the *equality* of the parts (units) is considered, and in (*c*) the *ratio* between the unit, $\$\frac{1}{2}$, and the whole, \$2, is stated.

Both (*a*) and (*b*) precede (*c*) in the order of thought. Without the idea of *equal* parts the comparison implied in (*c*) could not be made, and without taking cognizance of the *number* of parts, the ratio could not be defined. Hence (*c*) includes (*a*) and (*b*) as elements. Compare what has already been said with reference to the development of ratio.

It will be found valuable to write occasionally an expression on the blackboard, and get the pupils to interpret it as fully as they can. What does \$2 represent in the foregoing? What does \$$\frac{1}{2}$ represent? What does 4 show? What meaning is implied by not definitely expressed? Express clearly this implied meaning, etc.

Important Points.

(i) Lay stress on the *equality* of parts. Objects should be used freely at first.

(ii) Give many exercises in comparing denominate numbers. There should be much practical measurement.

(iii) Require the pupil to use each new fact as it is learnt. By this means it will become to him a source of power.

(iv) Let thought always precede expression. Figures should not be introduced too early.

(v) Require clear statements, both oral and written.

(vi) Get the pupil to illustrate his statements by means of objects, lines on blackboard, etc.

(vii) Ask the pupil frequently to interpret the written statements made by him.

(viii) Require the pupil to perform exercises in the shortest and most practical manner. For example, in adding $3\frac{1}{2}$, $18\frac{1}{4}$, $42\frac{5}{8}$, let the whole numbers and the fractions be added separately.

(ix) Give plenty of mental work in the form of practical problems.

Decimals.

(i) Introduce decimals by getting pupils to point out the effect of change of position on the value of digits, as 6, 66, 666, 6666, 8, 88, 8888, etc. Apply to 66/6, 66/66, 88/88, etc. (The

advantage of repeating the same digit will be seen at once.) These numbers should be expressed as 6 tens, 6 ones, 6 tenths, 6 hundredths, etc. Change to usual form as 66.66, etc.

(ii) Oral exercises.

(*a*) Read 3.5, 33.25, 423.625, etc.

(*b*) Express by figures 3 tens, 4 ones, and 5 tenths, etc.

(*c*) Find the value of .5 of \$10. Of .6 of \$50. Of .25 of \$80. Of .25 of 100 lbs., etc.

(*d*) Compare .2 and .4, .2 and .6, .2 and .8, .4 and .8, etc.

(iii) Reduction of decimal to vulgar fractions and *vice versa.*

(*a*) Represent a line 5 feet long on the blackboard. Divide it into foot lengths and bisect each of them. Give the class such work as: Point out $\frac{1}{5}$ of the line. Express this as a decimal. Point out .25 of it, .6 of it, etc. Express these as vulgar fractions.

(*b*) Divide the line into inches. How many inches make $\frac{3}{20}$ of the line? Express as a decimal, etc. Comparison. A great variety of valuable exercises may be given in this way. Their difficulty, too, may be increased to any degree by taking lines differing in length. In this work the idea of comparison should be made prominent.

(*c*) Represent a line $2\frac{1}{2}$ feet long. On it describe a square. Divide the square into 25 equal squares. Give work similar to the foregoing. Divide the square into equal parts in other ways and proceed as before.

(*d*) Write on blackboard such fractions as $\frac{2}{5}$, $\frac{3}{4}$, $\frac{5}{8}$, etc., and have pupils give their equivalents as decimals, etc. In case there should be doubt as to whether or not the pupil has thought out any relation which he has expressed, have him illustrate it objectively.

Addition and subtraction of decimal fractions are easily understood. Multiplication and division usually present difficulty in one particular, viz., finding the position of the decimal point in the product or the quotient as the case may be. Such exercises as these will direct attention to the main thing to be taught.

(*a*) Write on blackboard the expressions 5.5, 55.55, 12.335, 183.932, etc. Change the position of the decimal point. How is the value of the number affected, etc.?

(*b*) Multiply 642.937 by 10; by 100; by 1000. Divide the same by 10, etc. Give a sufficient number of exercises to familiarize the pupil with multiplying and dividing by powers of 10.

(*c*) Multiply 12.4 by 2, 20, 200, etc. Exercises.

(*d*) Multiply 125 by 4, .4, .04, .004, 4.4, 4.44, etc. Compare products, and account for their relations to one another.

(*e*) Multiply 12.5 by 4, .4, .04, .004, 4.4, etc. Exercises.

(*f*) Divide 44 by 20, 2, .2, .02, .002. Compare, and account for relations of quotients.

(*g*) Divide 4.4 by 2, 20, 200, 22, 2.2, .22, .022, etc. Comparison of quotients.

A large number of easy exercises related to one another should be given. The teacher must not depend entirely on

the textbook for these, because a textbook, no matter how much merit it may possess, cannot, in some cases at least, meet the wants of the individual.

It is important that the pupil should at first read decimals so as to bring out their full meaning. For instance, 72.345 should be read as seven tens, two ones, three tenths, four hundredths and five thousandths. After decimals are well known a briefer form of expression will be adopted, as seventy-two and three hundred and forty-five thousandths.

A common error in the school-room is to read such an expression as 7.125 as seven decimal one hundred and twenty-five. This must be guarded against.

PERCENTAGE.

It is often considered necessary to give a series of lessons leading up to percentage as if it were an entirely new division of arithmetic. Such is a waste of time, for the pupil who has a proper knowledge of fractions has really nothing new to learn except the meaning of the term; 3% means $\frac{3}{100}$ or .03, both of which are well known to the pupil. In fact no such phase of the subject would appear in our textbooks but for the fact that it is found convenient to adopt 100 as a standard on which to base certain arithmetical calculations. Percentage should therefore be taught in its relation to fractions, of which it forms a particular case.

The steps in teaching percentage may be taken as follows:

(i) Meaning of the term, *per cent.*

(*a*) Statement.

(*b*) Easy exercises to develop meaning of term, as what is 3% of \$100? Of \$200? Of \$400? Of 1000 bushels, etc.

(ii) Relation to decimals.

(*a*) Express 5%, 10%, 15%, 20%, 25%, 50%, etc., as decimals. Problems.

(*b*) Express .5, .55, .525, etc., as percentages. Problems.

(iii) Relation to vulgar fractions.

(*a*) Express 10%, 20%, 25%, 30%, 50%, etc., as vulgar fractions in their lowest terms. Problems.

(*b*) Express $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{50}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{3}{5}$, $\frac{13}{20}$, etc., as percentages. Problems.

The pupil should become thoroughly familiar with relations commonly used, as $\frac{1}{2}=.5=50\%$. By means of oral exercises, such as find the value of $\frac{4}{5}$ of $88, find .125 of $648, find $33\frac{1}{3}\%$ of $900, etc., the practical advantage of what is here stated may be shown. The fractions most important in this respect are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{10}$, $\frac{1}{20}$, $\frac{1}{25}$, and $\frac{1}{50}$.

(iv) General application. Types of problems.

(*a*) How much is 5% of $200? Of $50? Of $40? Of 90 gals.? Of 110? Of 420?

(*b*) Of what is each of the following 6%, $60, $72, 96 feet, 75 cords, 1200 bushels?

(*c*) $6 is what % of $12? Of $30? Of $480? Of $540?

(*d*) What sum increased by 5% of itself amounts to $105? $420? $1050?

(*e*) What sum diminished by 6% of itself equals $94? $470? $658?

(v) Particular applications.

The chief difficulty that will be experienced in dealing with the applications of percentage to commercial transactions will be in comprehending the transactions themselves. For

instance, if the pupil has not a clear idea of what Bank Discount is he may not be able to solve intelligently the simplest problem on it. Lessons should, therefore, be given on the business side of such subjects as interest, insurance, taxes, commission, banking, exchange, etc., in so far as such can be given in the school-room. The main thing, however, is for the pupil to get a firm grasp of the general principles underlying all of these. This being the case, he will be in a good position to deal with such subjects as the occasion arises in actual life.

Simple Interest and Bank Discount, on account of their practical importance, should receive special attention. Before arithmetical lessons are given on these the pupil must understand the nature of the transactions involved. To attempt to teach Bank Discount to pupils who have never seen a note, or who have no idea of what discounting a note means, is sheer folly. The pupils are simply working in the dark. But when the proper conditions are fulfilled the subjects here mentioned present almost no difficulty to anyone who has a good knowledge of the general application of percentage. The only new element involved in them is that of *time*.

Bank Discount.

The steps in teaching Bank Discount may be somewhat as follows :

(i) Let teacher and pupils have a short talk about lending money. As interest has already been taught, this will be familiar.

(ii) Discuss conditions under which money is lent.

(iii) Suggest a transaction and request pupils to write a promise to pay according to the given conditions.

(iv) Show them a promissory note in proper form. Question pupils on its chief conditions until they are understood.

(v) Have a talk about buying and selling notes. Let the teacher present a note drawn in his favour, say, for $100 due 6 months hence. What will it sell for if money is worth 6%? Exercises.

(vi) Introduce notes bearing interest in a similar manner. Exercises.

(vii) Find the face of a note when the selling price is given. The pupil should be able to work this out for himself. It is well, at first, to ask him to represent, by means of diagrams, the face of the note, the amount, the selling price, and the bank discount taken off by the purchaser.

The regular business method should be followed.

Compound Interest.

To the pupil who understands simple interest compound interest presents only one new idea, viz., that interest instead of being paid over to the lender when it becomes due may be retained by the borrower from period to period for a specified time at the same rate of interest as the original principal. As soon as the pupil understands this condition he should be given exercises to work out. No form of solution should be presented, neither should any explanations be offered, until the pupil has done all he can independently. The following will illustrate a method of dealing with the subject.

Find the compound interest on $500 for 2 years at 6%, payable annually.

Solution :

Principal for 1st year	=$500
	.06
Interest	=$ 30.00
	$500
	$530

Principal for 2nd year	=\$530
	.06
Interest	=\$ 31.80
	\$530
Amount at the end of 2 years	=\$561.80
	\$500
Interest for two years	=\$ 61.80

No objection can be taken to this form of solution as a means of giving the beginner a clear notion of what compound interest means and of how it may be calculated; but it is evident that after a few exercises are worked out the pupil cannot gain anything more except, perhaps, facility in addition and multiplication. But a great deal is still to be done if he is ever to master compound interest fully. He must consider such questions as these: What is the relation between \$500 and \$30? Account for this relation. Compare \$530 and \$500. Explain this. Compare \$530 and \$31.80. Compare \$561.80 and \$530. Why is this relation the same as that between \$530 and \$500? What must \$500 be multiplied by to give \$561.80? What relation does this bear to that between \$500 and \$530? etc. By this means the pupil will not only get excellent practice in comparing quantities, but he will arrive at a short method of finding compound interest, thus:

The amount $= \$500 \times (1.06)^2 = \$561.80$.

The interest $= \$561.80 - \$500 = \$\ 61.80$.

To be sure these statements might have been formed directly, but in that case they could not possibly mean as much to the pupil as they do after he has deduced them in the manner here indicated.

After the pupil has done this work he ought to be in a position to solve arithmetically such a problem as; The prin-

cipal is \$2000, the compound interest for 2 years is \$420; find the rate.

Solution :

$$\frac{\$2420}{\$2000} = \frac{1.21}{1.} = \text{Ratio when time is 2 years.}$$

$\therefore$ Ratio of amount for 1 year to Principal $= \sqrt{\frac{1.21}{1.}} = 1.1$

$\therefore$ Rate per unit $= .1$

$\therefore$ Rate per cent $= 10.$

Of course it will be seen that these exercises are suitable only for advanced pupils, for they demand considerable thought power.

Square Root.

During the early part of the course there should be given in connection with the work on factors and multiples, exercises on squaring small numbers, also on finding the square root of such numbers as 25, 36, 49, 81, 121, 144, etc.

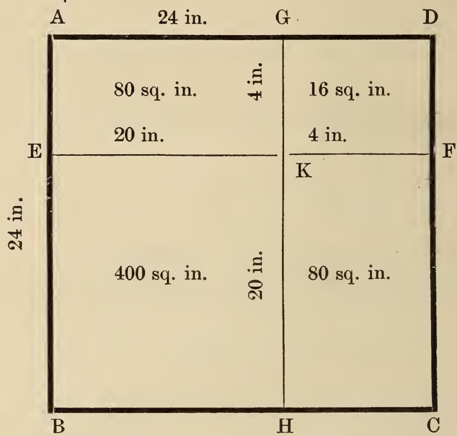
Before the formal process is taught the pupil should have a knowledge of terms and symbols; he should also know something of the application of square root in connection with such problems as these :

(i) The area of a square is 16 square feet; find its side.

(ii) The area of a rectangle whose length is twice its breadth is 288 square yards; find its length and breadth.

(iii) The length of a rectangular piece of ground is to its breadth as 5 : 3, its area is 2160 square rods; find its length and breadth.

Constructive exercises will make clear how the square of a number is made up. These may be carried on as follows: Describe on the blackboard a square whose side is 24 inches, thus:



$$
\begin{aligned}
\text{Area of square AC} &= \text{area of square EH} \\
&\quad + \text{area of rect. AK} \\
&\quad + \text{area of rect. CK} \\
&\quad + \text{area of square GF} \\
&= 400 \text{ sq. in.} + 80 \text{ sq. in.} \times 2 + 16 \text{ sq. in.} \\
&= 576 \text{ sq. in.}
\end{aligned}
$$

After the pupil has worked out a few problems of this kind his attention will be directed to the relation between the number of units of length and the number of units of surface. He will thus perceive that $24^2 = 20^2 + 20 \times 4 \times 2 + 4^2$. Great care must be taken here, or there may be confusion in the mind of the pupil with regard to the different kinds of units.

Simple exercises requiring the application of what has been learnt will follow, as: find the square of 15, 16, 18, 21, 23, 25, 27, 31, 32, 40, 41, 42, etc.

The next step is to present such numbers as can easily be separated into squares by inspection, as:

$$121=10^2+10\times1\times2+1^2 \therefore \sqrt{121}=11$$
$$144=10^2+10\times2\times2+2^2 \therefore \sqrt{144}=12$$
$$225=10^2+10\times5\times2+5^2 \therefore \sqrt{225}=15$$
$$400=20^2 \qquad\qquad \therefore \sqrt{400}=20$$
$$484=20^2+20\times2\times2+2^2 \therefore \sqrt{484}=22$$

The formal process should then be taught by applying it to numbers which have already been dealt with by the method of inspection.

$$\begin{array}{r}
10^2=100 \\
10\times6\times2=120 \\
6^2=\underline{\ \ 36} \\
256
\end{array}
\qquad
\begin{array}{r|l}
 & 256(10+6 \\
10^2= & \underline{100} \\
 & 156 \\
20\times6= & \underline{120} \\
 & \ \ 36 \\
6^2= & \ \ \underline{36}
\end{array}
\qquad
\begin{array}{r|l}
1 & 256(16 \\
 & \underline{1} \\
26 & 156 \\
 & \underline{156}
\end{array}$$

Review Exercises.

Among all the words found in the teacher's vocabulary none is more important than the word Review. Reviewing a subject does not mean merely repeating exercises previously performed. It includes far more than this, and serves a much higher purpose. When a particular section or division of a subject is first brought under consideration, attention must necessarily be confined to somewhat narrow limits, otherwise little progress could be made. The new, in order to be apperceived at all, must be related to the old, it is true, but we

must remember that the points of contact will be few, and the bonds of union will be weak. To pass on to the next division of the subject, and then to the next, and so forth in rapid succession would, under such conditions, result in the acquisition of a mass of unrelated or imperfectly related knowledge which would be of almost no value to its possessor. Knowledge gives power only when it is well organized.

The chief value of review work in arithmetic consists in giving the pupil a connected view of the subject in so far as he has studied it. Suppose, for example, that he has finished a series of exercises on percentage, he is now in a better position than ever before to comprehend the whole subject of fractions. He can see the advantage of having different fractional forms and, consequently, he can avail himself of this in their application. The pupil will take the greatest pleasure in such review lessons, for they are the means of representing to him old ideas in new relations.

Review exercises should consist chiefly of problems involving principles, and also facts which should be remembered, as tables, etc. Even in the regular daily exercises much time may be gained by requiring the pupil to recall incidentally such facts of arithmetic as are of practical importance in everyday life.